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A REMARK ON CHROMATIC POLYGROUPS**

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Polygroups are multivalued systems that satisfy group-like axioms. Natural examples, called chromatic polygroups, are obtained from certain edge colored complete graphs by making a multivalued algebra out of the set of colors. In [2] a close connection was given between polygroups and complete atomic integral relation algebras. Chromatic polygroups correspond to representable such algebras. Polygroups were also used to show that if permutation groups with certain properties exist, their orbital graphs must have a certain structure.

The main result of this paper is to establish a sufficient condition for a polygroup to be chromatic. All polygroups with at most 3 elements are chromatic. There are exactly 102 (isomorphism) types of 4 element systems ([1]) of which at least 28 are known not to be chromatic. The condition in THEOREM 1 applies to at least 30 of the 4 element systems. Fortunately, a stronger criteria, presented in [2], works for at least 15 of these. In the proof of THEOREM 2 we show in detail how to verify the condition in THEOREM 1 for the system N_1.

We conclude the paper by showing that although N_1 is a chromatic polygroup, it is not isomorphic to a polygroup obtainable from double cosets of a finite group. Such an example is closely related to questions raised by Ralph McKenzie in [4] with regard to permutational relation algebras.

1. PRELIMINARIES

We recall the following definitions from [2].

A polygroup is a system $\mathcal{M} = \langle M, \cdot, e, ^{-1} \rangle$ where $e \in M$, $^{-1}$ is a unary operation on M, $\cdot$ maps M^2 into nonempty subsets of M, and the following axioms hold for all x,y,z in M:

(P_1) $\quad (x\cdot y)\cdot z = x\cdot(y\cdot z)$,

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(P_2) $x \cdot e = x = e \cdot x$,

(P_3) $x \varepsilon y \cdot z$ implies $y \varepsilon x \cdot z^{-1}$ and $z \varepsilon y^{-1}$.

Many important polygroups are derived from color schemes, a notion that extends D.G.Higman's homogeneous coherent configuration (see [3]). Suppose $\mathcal{C}$ is a set(of colors) and ϵ is an involution of $\mathcal{C}$. A <u>color scheme</u> is a system $\mathcal{V} = \langle V, C_a \rangle_{a \varepsilon \mathcal{C}}$ where

(i) $\{C_a : a \varepsilon \mathcal{C}\}$ is a partition of $V^2 - I = \{(x,y) \varepsilon V^2 : x \neq y\}$,

(ii) $C_a^{\vee} = C_{\epsilon(a)}$ for all $a \varepsilon \mathcal{C}$,

(iii) for each color and each vertex the color is present on some edge from the vertex,

(iv) for $a,b,c \varepsilon \mathcal{C}$ if $(x,y) \varepsilon C_c$, the existence of an (a,b)-path from x to y is independent of the choice of x and y, in symbols, $C_c \cap (C_a | C_b) \neq \emptyset \Rightarrow C_c \subseteq C_a | C_b$.

$\mathcal{V}$ is called a <u>partial scheme</u> it it satisfies (i) and (ii). Given a color scheme $\mathcal{V}$, choose a new symbol $I \notin \mathcal{C}$. (Think of I as the identity relation.) The <u>algebra</u>(<u>color algebra</u>, <u>configuration algebra</u>) of $\mathcal{V}$ is the system $M_{\mathcal{V}} = \langle \mathcal{C} \cup \{I\}, \cdot, ^{-1}, I \rangle$ where $a^{-1} = \epsilon(a)$ for $a \varepsilon \mathcal{C}$, $I^{-1} = I$,

$$x \cdot I = x = I . x \quad \text{for all} \quad x \varepsilon \mathcal{C} \cup \{I\}, \text{and}$$

for $a,b,c \varepsilon \mathcal{C}$,

$$a \cdot b = \{c \varepsilon \mathcal{C} : C_c \subseteq C_a | C_b\} \cup \{I : b = a^{-1}\}.$$

A polygroup isomorphic to the algebra of a color scheme is called <u>chromatic</u>.

A natural example of a chromatic polygroup is the system $G /\!/ H$ of all double cosets of a group G modulo a subgroup H. Namely,

$$G /\!/ H = \langle \{HgH : g \varepsilon G\}, \cdot, H, ^{-1} \rangle$$

where $(HgH) \cdot (Hg'H) = \{Hghg'H : h \varepsilon H\}$ and $(HgH)^{-1} = (Hg^{-1}H)$. This construction generalizes to that of a (double)quotient. An equivalence relation θ on a polygroup M is called a (<u>full</u>) <u>conjugation</u> on M if

(i) $x \theta y$ implies $x^{-1} \theta y^{-1}$,

(ii) $z \varepsilon x \cdot y$ and $z' \theta z$ implies there exist $x' \theta x$ and $y' \theta y$ such that $z' \varepsilon x' \cdot y'$.

The natural quotient system, denoted $M /\!/ \theta$, is a polygroup. In [2] it is shown that $Q^2(\text{Group}) = \{G/\!/\theta : \theta$ is a full conjugation on some group $G\}$ is a class of chromatic polygroups.

2. A SUFFICIENT CONDITION

A simple condition on a polygroup was given in [2] for the existence of a random color scheme whose algebra is the given polygroup. This condition is generalized in THEOREM 1. As with the previous condition we use the fact a polygroup can be described by a collection of "forbidden configurations".

Suppose $\mathcal{V}$ is a color scheme with a set $\mathcal{C}$ of colors. An element of $\mathcal{C}^3$ is called a colored triangle or just triangle. A triangle (i,j,k) is realizable in $\mathcal{V}$ on an edge $(x,y)\varepsilon V^2-I$ if $(x,y)\varepsilon C_k \cap (C_i | C_j)$. A class K of colored triangles is locally realizable in $\mathcal{V}$ if, for every edge $(x,y)\varepsilon C_k$ and every $(i,j,k)\varepsilon K$, $(x,y)\varepsilon C_i | C_j$. A class K of colored triangles is forbidden in $\mathcal{V}$ if no triangle in K is realizable on an edge in $\mathcal{V}$. Note that condition (iv) in the definition of color scheme means that if a colored triangle is not forbidden it is locally realizable. A polygroup $\mathcal{M}$ is completely determined by the class of colored triangles

$$FC(\mathcal{M})=\{(a,b,c)\varepsilon(M^-)^3 : c\notin a\cdot b\}$$

called its forbidden class. [M^- is $M-\{e\}$.] $\mathcal{M}$ is chromatic iff there exist a partial color scheme that forbids $FC(\mathcal{M})$ and locally realizes its complement.

We only consider classes of triangles that are closed under the obvious symmetries; thus, for example, if $(i,j,k)\varepsilon K$, then so does $(k,\epsilon(j),i)$ and $(\epsilon(i),k,j)$. Occasionally this saves writing, for example, the polygroup N_1 given in TABLE 1 has a forbidden class of 11 elements, however, $FC(N_1)$ is "generated" by $\{(1,1,2),(1,1,3),(1,3,1)\}$. With the terminology above the main result can be given.

THEOREM 1. A finite polygroup $\mathcal{M}$ is chromatic if $K=FC(\mathcal{M})$ has the property:

(*) for every partial scheme $\mathcal{V}=\langle V,C_a\rangle_{a\varepsilon\mathcal{C}}$ that forbids K, every edge $(x,y)\varepsilon C_k$ and every triangle $(i,j,k)\notin K$, there exist a partial scheme $\mathcal{V}'$ extending $\mathcal{V}$ such that $\mathcal{V}'$ forbids K and (i,j,k) is realized on (x,y) in $\mathcal{V}'$.

Proof. A color scheme is constructed by induction. To begin, choose a trivial partial scheme V_0 that forbids K, eg., an edge or some admissible triangle. By induction we construct a chain $V_0 \subseteq V_1 \subseteq \ldots$ of finite partial schemes such that each V_{n+1} forbids K but realizes all triangles not in K on edges in V_n. The union of this chain is then a partial scheme that forbids K and locally realizes its complement. Hence it is a color scheme whose algebra is the given M. Now, to construct V_{n+1} from V_n enumerate all edges in V_n and construct a finite chain of finite extensions of V_n, one for each edge. To construct an extension corresponding to an edge (x,y) enumerate all triangles not in K which need to be realized on (x,y). We apply (*) repeatedly (once for each triangle) to build an extension of V_n that realizes each of these triangles (and also all triangles realized on "previous" edges). □

As will be seen in the following section it is more difficult to verify (*) than the stronger condition given in 5.3 of [2]. Also, we have less control over the final color scheme in the construction above. For example, we do not know when the resulting scheme is homogeneous nor when its automorphism group is transitive on vertices. THEOREM 3 below strongly suggests there are color schemes with no polygroup equivalent schemes having these properties.

The condition (*) in THEOREM 1 is probably not necessary for M to be chromatic. A possible example is the polygroup A^{01}_{14} whose forbidden class is $\{(1,1,2),(1,1,3),(2,2,2),(3,3,3)\}$. It is not known whether this system is chromatic. However, it can be shown that if it is chromatic, the C_1 monochrome subgraph of a representing scheme is 3-partite and so there are partial schemes that fail to satisfy (*).

3. TWO NONCOMMUTATIVE CHROMATIC POLYGROUPS

The multiplication tables for the systems N_1 and N_5 are given in TABLE 1. Using THEOREM 1 we show

THEOREM 2. N_1 and N_5 are chromatic polygroups.

This is the most difficult step needed for the following.

THEOREM 3. There are chromatic polygroups not isomorphic to a double coset system G//H for any finite groups G and H.

Proof. N_1 and N_5 are two such polygroups. They are chromatic by the result above. The following argument, pointed out by P.J.Cameron, shows that a double coset algebra $G//H$ with 4 elements where G is a finite group must be commutative. A finite group G induces a coherent configuration on the coset space of H. The corresponding centralizer algebra A is semisimple and decomposes into matrix rings $A_i = M(\mathbb{C}, e_i)$ (see [3]). If $G//H$ has 4 elements, A has dimension 4 and A_0 has dimension 1; so $4 = 1 + \Sigma e_i^2$. It follows that each A_i has dimension 1 so A (and hence $G//H$) is commutative. □

We conclude the paper with the long and tedious proof of THEOREM 2.

Proof. (THEOREM 2) We only consider N_1, the details for N_5 are similar. It suffices to show that $FC(N_1)$ satisfies (*). To aid the discussion we classify points of a partial scheme relative to a given edge. For $x,y \in V$, $x \neq y$, we say $z \in V$, $z \neq x,y$ has type (α,β) with respect to (x,y) if $(x,z) \in C_\alpha$ and $(z,y) \in C_\beta$. Now, suppose $(x,y) \in V^2 - I$, $(i,j,k) \notin K$, and (x,y) is in C_k. If there exist $z \in V$ with type (i,j), we can set $V' = V$ and we are done. Assume V contains no element of type (i,j) with repsect to (x,y). Choose $v_{ij} \notin V$ and set

$$V' = V \cup \{v_{ij}\}.$$

Extend the coloring C_1, C_2, C_3 of V to the new edges of V' as follows: (We only treat (v_{ij},z) since edges (z,v_{ij}) are assigned to the "paired" color.)

(1) $(v_{ij},x) \in C'_{\epsilon(i)}$ and $(v_{ij},y) \in C'_j$,

(2) for $z \neq x,y$ the color assigned to (v_{ij},z) depends on i,j,k;

(i) if $(i,j,k)=(1,1,1)$ or $(2,2,2)$, put $(v_{ij},z) \in C'_1$ if z has type $(1,2)$ and in C'_2 if z has type $(2,1)$.

(ii) if $(i,j,k)=(2,1,1)$ or $(3,1,1)$, put $(v_{ij},z) \in C'_1$ for z with type $(1,1)$ or type $(1,2)$.

(iii) if $(i,j,k)=(2,1,2)$ or $(2,3,2)$, put $(v_{ij},z) \in C'_1$ for z with type $(2,2)$ or $(1,2)$.

(iv) if $(i,j,k)=(1,2,1)$, $(1,2,2)$, or $(1,2,3)$, put $(v_{ij},z) \in C'_2$ for all z.

(v) if $(i,j,k)=(1,3,3)$, put $(v_{ij},z)\varepsilon C_1'$ for z with type $(1,2)$ and in C_2' for z with type $(2,3)$.

(vi) if $(i,j,k)=(3,2,3)$, put $(v_{ij},z)\varepsilon C_1'$ for z with type $(1,2)$ and in C_2' for z with type $(3,1)$.

(vii) if $(i,j,k)=(3,1,3)$, put $(v_{ij},z)\varepsilon C_1'$ for z with type $(1,2)$ or $(3,2)$.

(viii) if $(i,j,k)=(2,3,3)$, put $(v_{ij},z)\varepsilon C_1'$ for z with type $(1,2)$ or $(1,3)$.

(ix) put $(v_{ij},z)\varepsilon C_3'$ in all other cases.

The definition above gives a partial scheme $\mathcal{V}'=\langle V',C_1',C_2',C_3'\rangle$. This scheme clearly realizes the given triangle (i,j,k) on (x,y) so it remains to show that $\mathcal{V}'$ forbids K. Suppose $\{u,v,w\}$ determines a triangle in $\mathcal{V}'$. We may assume one of the vertices is v_{ij} since $\mathcal{V}$ forbids K; moreover, since (i,j,k) is not forbidden, we may assume at least one of the vertices is not x or y. This leaves three possible cases:

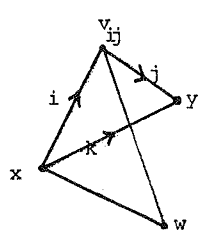
(a) $\{u,v,w\}=\{v_{ij},x,w\}$ where $w\varepsilon V\sim\{x,y\}$;

(b) $\{u,v,w\}=\{v_{ij},y,w\}$ where $w\varepsilon V\sim\{x,y\}$;

(c) $\{u,v,w\}=\{v_{ij},v,w\}$ where $v,w\varepsilon V\sim\{x,y\}$.

We show the triangles realized in $\mathcal{V}'$ in each of these cases are not in K.

<u>case (a)</u>.

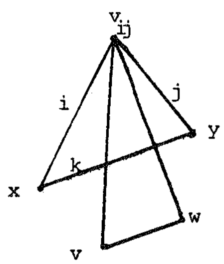


Suppose $i=1$. If $(v_{ij},w)\varepsilon C_1'$, w has type $(1,2)$ with respect to (x,y) so the realized triangle is $(1,1,1)$ which is not in K. No triangle in K can be produced if $(v_{ij},w)\varepsilon C_2'$; and if $(v_{ij},w)\varepsilon C_3'$, $(x,w)\varepsilon C_3$ which again precludes any triangle in K. Now suppose $i=2$. If $(v_{ij},w)\varepsilon C_1'$ then (x,w) is never in C_3; so a triangle in K is not produced. Likewise if

$(v_{ij},w)\varepsilon C_2'$ then $(x,w)\varepsilon C_2$ and if $(v_{ij},w)\varepsilon C_3'$ then (x,w) is never in C_1; so again these conditions rule out producing a triangle in K. Finally suppose i=3. If $(v_{ij},w)\varepsilon C_3'$ the triangle cannot belong to K. If (v_{ij},w) is in C_1' then (x,w) is never in C_2 and if the edge is in C_2' then (x,w) belongs to C_3. In either case no triangle in K results. Thus, we have seen above that no triangle in K is introduced by means of u,v,w in case (a).

<u>case (b)</u>. This case for a given (i,j,k) is the same as case (a) for triangle $(\epsilon(j),\epsilon(i),\epsilon(k))$, so no triangle in K can be realized in this situation.

<u>case (c)</u>.



We first consider the case where both (v_{ij},v) and (v_{ij},w) are assigned the same color, say C_r. No triangle in K is possible if r=2 or r=3. In case r=1 the only way to produce an element of K would be for (v,w) to belong to C_3. This never happons when v,w have one of the types: (1,2), (2,2), or (3,2). If either v or w have type (1,1), then (by clause (ii) defining the assignment) one of v or w has type (1,1) while the other has type (1,1) or (1,2). In either case (x,v), $(x,w)\varepsilon C_1$, so $(v,w)\not\varepsilon C_3$. If either v or w has type(1,3), then (by clause (viii)) (x,v), $(x,w)\varepsilon C_1$. Since $\{v,x,w\}$ determines a (2,1,-) triangle in V, $(v,w)\not\varepsilon C_3$. Hence no triangle in K can be produced when the same colors are assigned to both (v_{ij},v) and (v_{ij},w).

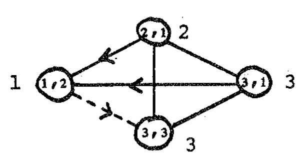
Now suppose these edges are assigned different colors. In place of writing out the details as above we present diagrams which reveal the forced relationships among the types of elements (with respect to (x,y)) and v_{ij}. We refer to this graph as a diagram for v_{ij}^k. A vertex (α,β) in the diagram denotes a point in V with type (α,β) with respect to (x,y). The lable r next to such a vertex indicates the edge (v_{ij},z)

is in C_r for all z with the indicated type. An edge between two vertices of the diagram indicates the way in which the edges between the corresponding types are colored. A solid edge means the corresponding types are connected by a C_3-edge, an arrow ⟶ indicates a C_1-edge (the reverse direction is automatically in C_2), and a dashed arrow - - -> - - means $C_2 \cup C_3$ (i.e.,all except C_1). For example, in the v_{11}^1 diagram below, if u has type (2,1) and v has type (1,2), the edge (u,v) must be in C_1 while $(v,u) \varepsilon C_2$. This is easily seen since x has type (1,1)with respect to the edge (u,v). This forces $(u,v) \varepsilon C_1$ since triangles (1,1,2) and (1,1,3) are forbidden in V. If w has type (3,3), (v,w) is not completely forced; it may be either in C_2 or C_3 but it is forbidden to be in C_1. We now return to the proof.

If k=1 there are three possibilities for (i,j):

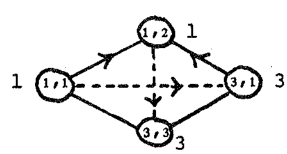
(1,1), (2,1), (3,1).

[All new edges are assigned to C_2' in case (i,j)=(1,2) and all are in C_3' if (i,j)=(3,3)]. To check whether a triangle in K is introduced in the (1,1) case construct the v_{11}^1 diagram.

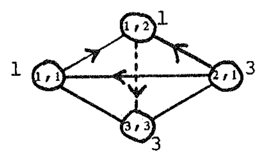


v_{11}^1 diagram

First observe that the assignments will not produce a triangle in K if $(v_{ij},v)\varepsilon C_1'$ and (v_{ij},w) is one of the other types. Similarly, if $(v_{ij},v)\varepsilon C_2'$ and $(v_{ij},w)\varepsilon C_3'$ the triangles that result are not in K. The same conclusion can be verified for the other two choices of (i,j) using the diagrams for v_{21}^1 and v_{31}^1.

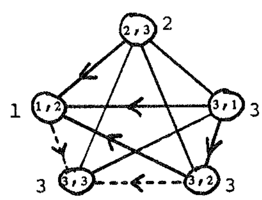


v_{21}^1 diagram

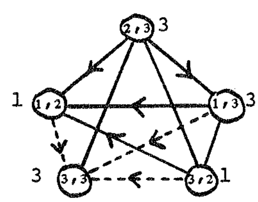


v^1_{31} diagram

The case where k=2 is dual to the k=1 case. For k=3 there are two cases (up to symmetry). The diagrams of relationships are given below.



v^3_{13} $(=v^3_{32})$ diagram



v^3_{31} $(=v^3_{23})$ diagram

From the diagrams above we can check that no triangles in K are introduced by case (c) using the assignments of V'. Hence we conclude that V' forbids K which completes the verification of (*) for N_1. Thus by THEOREM 1 N_1 is chromatic.

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TABLE 1

N_1	0	1	2	3
0	0	1	2	3
1	1	1	0123	3
2	2	012	2	23
3	3	13	3	0123

N_5	0	1	2	3
0	0	1	2	3
1	1	123	0123	23
2	2	012	123	123
3	3	123	13	0123