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The representation of 3–dimensional cylindric algebras

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When the representation of a cylindric algebra (a CA) is mentioned, normally one thinks about representing the algebra as a subdirect product of cylindric set algebras. The class of CA_α's (cylindric algebras of dimension α) having such a (set) representation is denoted by RCA_α. However, there are other constructions which can be used to represent CA's. For example, in [8], 2.7.43, every CA is embedded into the complex algebra of a relational system of a specific type called a cylindric atom structure. The term "relational representability" is used in [8] to refer to this type of representation. Actually, the notion of representability in terms of set algebras is a form of relational representability using a special class of atom structures (cf., 2.7.45 of [8]). Another special type of relational representation was considered in Monk [12] where integral CA_3's were associated with quasigroups. This construction was used to produce CA_3's that are not in RCA_3. The starting point for the ideas presented here was the problem posed in [12] about whether every integral CA_3 is isomorphic to the complex algebra of a quasigroup. The answer is shown to be "no" in Theorem 4.4, but if, in Monk's construction, "quasigroup" is replaced by "multi–valued loop" and "duplicate atoms" are allowed, the answer becomes "yes" (Theorem 3.4(iv)). Essentially the same argument allows every complete and atomic CA_3 to be "coordinatized" by a certain kind of multi–valued groupoid (Theorem 3.4(i)).

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After a few preliminaries about CA_3's in Section 1, various multi–valued systems are discussed in Section 2. The main representation results are given in Section 3. In Section 4 various classes of CA_3's are characterized using the multi–valued representations and the problems raised in Monk [12] are answered. In particular, it is shown that the class of group representable CA_3's is not finitely axiomatizable over the class of integral CA_3's. This is a CA_3 analogue of McKenzie's nonfinite axiomatizability result [11] for group representable relation algebras. The representation of integral CA_3's by multi–valued loops and the analogue of McKenzie's work were announced in [2] and [4] respectively.

1. Cylindric algebras of dimension 3

In this section we recall notation and terminology about cylindric algebras and establish preliminary results needed in section 3.

A cylindric algebra of dimension 3, or CA_3, is a system

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_1, d_{ij} \rangle_{ij<3}$$

that satisfies the following conditions for all $i, j, k < 3$ and all $x, y \in A$:

(C_0) $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra,
(C_1) $c_i 0 = 0$,
(C_2) $x \leq c_i x$,
(C_3) $c_i(x \cdot c_i y) = c_i x \cdot c_i y$,
(C_4) $c_i c_j x = c_j c_i x$,
(C_5) $d_{ii} = 1$,
(C_6) $c_j(d_{ij} \cdot d_{jk}) = d_{ik}$ if $j \neq i, k$,
(C_7) $c_i(d_{ij} \cdot x) \cdot c_i(d_{ij} \cdot -x) = 0$ if $i \neq j$.

The algebraic theory of cylindric algebras has been extensively developed in Henkin, Monk, and Tarski [8] and [9]. For a brief readable introduction to the theory, see Monk [12]. The notation and terminology of [8] and [9] will be used with the following exception: if R is a relation and $X \cup \{x\}$ is a subset of the domain of R, the R–image of X is denoted RX (or $R(X)$ if clarity is needed) instead of R^*X and the R–image of x is denoted Rx (or $R(x)$) instead of R^*x. In particular, for an equivalence relation θ, the equivalence class of x, $x/\theta(= \theta^* x)$ is denoted by θx.

The property in Lemma 1.1(i), given below, was introduced in Monk [12]. A CA_3 that satisfies this property is called *integral*. It is obvious that every non–trivial integral CA_3 is simple (cf., 2.3.14 of [8]).

Lemma 1.1. *In a non–trivial $CA_3\mathfrak{A}$ the following are equivalent:*

(i) $\forall x(x \neq 0 \Longrightarrow c_0c_1x = c_0c_2x = c_1c_2x = 1)$,
(ii) $d_{01} \cdot d_{12}$ *is an atom.*

Proof. (i) $\Longrightarrow$ (ii). Suppose (i) holds and assume $x, y \neq 0$, $x \cdot y = 0$, and $x + y = d_{01} \cdot d_{12}$ in $\mathfrak{A}$. Use (C_7) twice and the fact $c_0x + c_0y = d_{12}$ to obtain $c_1c_0x \cdot c_1c_0y = 0$. So, not both c_0c_1x and c_0c_1y can equal 1 which contradicts (i). Thus, either $d_{01} \cdot d_{12}$ is an atom or $d_{01} \cdot d_{12} = 0$. But, $d_{01} \cdot d_{12} = 0$ implies $\mathfrak{A}$ has only one element; so (ii) holds. (ii) $\Longrightarrow$ (i). If $c_0c_1x \cdot d_{01} \cdot d_{12} = 0$, then

$$0 = c_0c_10 = c_0c_1(c_0c_1x \cdot d_{01} \cdot d_{12}) = c_1(c_0c_1x \cdot d_{12}) = c_0c_1x$$

which implies $x = 0$. Thus, if (ii) holds and $x \neq 0$, then $d_{01} \cdot d_{12} \leq c_0c_1x$ which yields $1 = c_0c_1(d_{01} \cdot d_{12}) = c_0c_1x$. The other cases are similar. ∎

A CA_3 is called complete and atomic if its Boolean part is complete and atomic. By 2.7.20 of [8] every CA_3 can be embedded in a complete and atomic CA_3 (its perfect extension). This embedding preserves various properties including simplicity (2.7.17 of [8]). An argument similar to 2.7.17 shows that being integral is also preserved.

Lemma 1.2. *Every (integral) CA_3 can be embedded in an (integral) CA_3 that is complete and atomic.*

Complete and atomic CA's can be studied by structures on the atoms. The *cylindric atom structure* of a complete and atomic CA_3 $\mathfrak{B}$ (cf., 2.7.32 of [8]) is the system $\mathfrak{At}\mathfrak{A} = \langle At\mathfrak{B}, T_i, E_{ij}\rangle_{i,j<3}$ where $T_i = \{(x, y) \in At\mathfrak{B} \times At\mathfrak{B} : c_ix = c_iy\}$ and $E_{ij} = \{x \in At\mathfrak{B} : x \leq d_{ij}\}$. The next lemma axiomatizes the notion of an cylindric atom structure.

Lemma 1.3. *(2.7.40–2.7.41 of [8]) The cylindric atom structure of a complete atomic CA_3 can be characterized as a relational system $\mathfrak{B} = \langle B, T_i, E_{ij}\rangle_{i,j<3}$ where $T_i \subseteq B \times B$, $E_{ij} \subseteq B$ such that for all $i, j, k < 3$:*

(i) *T_i is an equivalence relation on B,*
(ii) $T_i|T_j = T_j|T_i$,
(iii) $E_{ij} = B$,
(iv) $E_{ij} = T_k(E_{ik} \cap E_{kj})$ *for $i, j \neq k$,*
(v) $|T_ix \cap E_{ij}| = 1$ *whenever $x \in B$, $i \neq j$.*

Every cylindric atom structure determines a complete atomic CA_3. Namely, from such a structure $\mathfrak{B} = \langle B, T_i, E_{ij} \rangle_{i,j<3}$ we obtain a complete atomic CA_3:

$$\mathfrak{Cm}\mathfrak{B} = \langle SbB, \cup, \cap, \sim, \emptyset, B, c_i, E_{ij} \rangle_{i,j<3}$$

where, for $X \subseteq B$ and $i < 3$, $c_i X = \{y \in B : xT_i y \text{ for some } x \in X\}$. $\mathfrak{Cm}\mathfrak{B}$ is called the *complex algebra* of $\mathfrak{B}$ (cf., 2.7.33 of [8]). It is easily seen that every complete atomic CA_3 $\mathfrak{A}$ is isomorphic to $\mathfrak{Cm}\mathfrak{At}\mathfrak{A}$ (2.7.43 of [8]).

The connection between complete atomic CA's and cylindric atom structures allows CA concepts to be introduced via relational systems. This approach is used below to introduce the notion of an adjunction of a complete atomic CA_3, that is, an extension of the algebra that is obtained by replacing one or more atoms $x \not\leq d_{01} + d_{12} + d_{02}$ by a set of atoms each of which act like x. Such an extension can be produced by an appropriate iteration of dilations (cf., 3.2.69 of [9]).

Definition 1.4.

(i) *For cylindric atom structures* $\mathfrak{B} = \langle B, T_i, E_{ij} \rangle_{i,j<3}$ *and* $\mathfrak{B}' = \langle B', T'_{ij}, E'_{ij} \rangle_{i,j<3}$ *a function* h *from* B *onto* B' *is a full homomorphism of* $\mathfrak{B}$ *onto* $\mathfrak{B}'$*, in symbols* $h : \mathfrak{B} \succ \mathfrak{B}'$*, if, for all* $x, y \in B$ *and* $i, j < 3$,

$$xT_i y \Leftrightarrow (hx)T'_i(hy) \quad \text{and} \quad x \in E_{ij} \Leftrightarrow (hx) \in E'_{ij}\,.$$

(ii) *For complete and atomic* CA_3*'s* $\mathfrak{A}$ *and* $\mathfrak{B}$*,* $\mathfrak{A}$ *is an adjunction of* $\mathfrak{B}$*, in symbols* $\mathfrak{A} \in \mathbf{Adj}(\mathfrak{B})$*, if* $h : \mathfrak{At}\mathfrak{A} \succ \mathfrak{At}\mathfrak{B}$ *for some full homomorphism* h*. If* K *is a class of* CA_3*'s, let* $\mathbf{Adj}(K) = \cup\{\mathbf{Adj}(\mathfrak{B}) : \mathfrak{B} \in K\}$.

Proposition 1.5.

(i) *If* $\mathfrak{B}$ *and* $\mathfrak{B}'$ *are cylindric atom structures and* $\mathfrak{B} \succ \mathfrak{B}'$*, then* $\mathfrak{Cm}\mathfrak{B}'$ *is embeddable in* $\mathfrak{Cm}\mathfrak{B}$.

(ii) *For* $\mathfrak{A}$ *and* $\mathfrak{B}$ *complete atomic* CA_3*'s,* $\mathfrak{A} \in \mathbf{Adj}(\mathfrak{B})$ *implies* $\mathfrak{B}$ *is isomorphic to a subalgebra of* $\mathfrak{A}$.

Proof. (i) If $\mathfrak{B} \succ \mathfrak{B}'$, there exist a full homomorphism h of $\mathfrak{B}$ onto $\mathfrak{B}'$. It is straight forward to check that the map

$$x \longmapsto \cup\{h^{-1}a : a \in x\}$$

is an embedding of $\mathfrak{Cm}\mathfrak{B}'$ into $\mathfrak{Cm}\mathfrak{B}$.

(ii) From (i) and the fact that for every complete atomic CA_3 $\mathfrak{A}$, $\mathfrak{A} \cong \mathfrak{Cm}\mathfrak{At}\mathfrak{A}$. ∎

For complete atomic CA_3 's $\mathfrak{A}$ and $\mathfrak{B}$ we say that $\mathfrak{B}$ is *minimal with respect to* $\mathfrak{A} \in \mathbf{Adj}(\mathfrak{B})$ if $\mathfrak{C} \in \mathbf{Adj}(\mathfrak{B})$ for every complete atomic $\mathfrak{C}$ with $\mathfrak{A} \in \mathbf{Adj}(\mathfrak{C})$. In particular, it follows that $\mathfrak{A} \in \mathbf{Adj}(\mathfrak{B})$.

If θ is an equivalence relation on $At\mathfrak{A}$, we let $\mathfrak{At}\mathfrak{A}/\theta$ denote the *quotient* of the relational system $\mathfrak{At}\mathfrak{A}$ by θ; that is, $\mathfrak{At}\mathfrak{A}/\theta$ is the system $\langle At\mathfrak{A}/\theta, T_i', E_{ij}'\rangle$ where $At\mathfrak{A}/\theta = \{\theta x : x \in At\mathfrak{A}\}$ is the collection of all θ–blocks, $(\theta x)T_i'(\theta y) \Leftrightarrow xT_iy$, and $\theta x \in E_{ij}' \Leftrightarrow x \in E_{ij}$ for all $i, j < 3$.

Proposition 1.6. *Suppose $\mathfrak{A}$ is a complete atomic CA_3 with cylindric atom structure $\mathfrak{At}\mathfrak{A} = \langle At\mathfrak{A}, T_i, E_{ij}\rangle_{i,j<3}$ and $\theta = T_0 \cap T_1 \cap T_2$. Then*

(i) *$\mathfrak{At}\mathfrak{A}/\theta$ is a cylindric atom structure;*
(ii) *$\mathfrak{Cm}\mathfrak{At}\mathfrak{A}/\theta$ is isomorphic to a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ that is minimal with respect to $\mathfrak{A} \in \mathbf{Adj}(\mathfrak{B})$;*
(iii) *The subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ in (ii) is unique. In fact it is characterized by*

$$x \in B \quad \text{iff} \quad \forall a \in At\mathfrak{A}(a \leq x \Longrightarrow c_0a \cdot c_1a \cdot c_2a \leq x)\,.$$

Proof. (i). Clearly θ is an equivalence relation on $At\mathfrak{A}$ such that for all $x, y \in At\mathfrak{A}$

$$x\theta y \quad \text{iff} \quad c_ix = c_iy \quad \text{for all} \quad i < 3\,.$$

Properties 1.3(i)–(iii) are obvious for $\mathfrak{At}\mathfrak{A}/\theta$. For 1.3(iv),

$$\begin{aligned}\theta x \in E_{ij}' &\Leftrightarrow x \in E_{ij} \Leftrightarrow xT_ky \text{ for some } y \in E_{ik} \cap E_{jk} \text{ (1.3(iv) for } \mathfrak{At}\mathfrak{A})\\ &\Leftrightarrow (\theta x)T_k'(\theta y) \text{ for some } \theta y \in E_{ik}' \cap E_{kj}'\\ &\Leftrightarrow \theta x \in T_k'(E_{ik}' \cap E_{kj}')\,.\end{aligned}$$

For 1.3(v), suppose $\theta x, \theta y \in E_{ij}'$ with $i \neq j$ and $(\theta x)T_i'(\theta y)$. Then $x, y \in E_{ij}$ and xT_iy so $x = y$ by 1.3(v) for $\mathfrak{At}\mathfrak{A}$. This finishes the proof of (i).

(ii). The map $\theta^* : x \longmapsto \theta x$ shows that $\mathfrak{At}\mathfrak{A} \succ /\mathfrak{At}\mathfrak{A}\,\theta$. Thus, by 1.5(i) $\mathfrak{Cm}\mathfrak{At}\mathfrak{A}/\theta$ is isomorphic to a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. Now, if $\mathfrak{A} \in \mathbf{Adj}(\mathfrak{C})$ for some complete atomic $\mathfrak{C}$, there is a full homomorphism $h : \mathfrak{At}\mathfrak{A} \succ \mathfrak{At}\mathfrak{C}$. For $x, y \in At\mathfrak{A}$, $hx = hy$ implies $hxT_i'hy$ and hence xT_iy for all i. Thus, $\ker h \subseteq \theta$. Hence there exist a full homomorphism $k : \mathfrak{At}\mathfrak{B} \succ \mathfrak{At}\mathfrak{A}/\theta$ such that $\theta^* = k \circ h$. Therefore, $\mathfrak{C} \in \mathbf{Adj}(\mathfrak{Cm}\mathfrak{At}\mathfrak{A}/\theta)$. Thus, $\mathfrak{Cm}\mathfrak{At}\mathfrak{A}/\theta$ is minimal with the property.

(iii). Suppose $\mathfrak{B}$ is the subalgebra of $\mathfrak{A}$ constructed in (ii). Since $\mathfrak{At}\mathfrak{B} \cong \mathfrak{At}\mathfrak{A}/\theta$, the atoms of $\mathfrak{B}$ are elements $\Sigma^{\mathfrak{A}}\theta x$ where $x \in At\mathfrak{A}$ and $\theta = T_0 \cap T_1 \cap T_2$. Then

$$(1)\ c_0x \cdot c_1x \cdot c_2x = c_0(\Sigma\theta x) \cdot c_1(\Sigma\theta x) \cdot c_2(\Sigma\theta x) = \Sigma\theta x \quad \text{for every} \quad x \in At\mathfrak{A}.$$

First note that $c_i(\Sigma\theta x) = \Sigma c_i\theta x = c_ix$ for all i. Now, suppose $x, y \in At\mathfrak{A}$ and $y \leq c_0x \cdot c_1x \cdot c_2x$. Then $c_iy \leq c_ix$ and since c_ix is an atom of the BA of c_i–closed elements of $\mathfrak{A}$, $c_iy = c_ix$. Hence, $x\theta y$ and $y \leq \Sigma\theta x$. Thus, $c_0x \cdot c_1x \cdot c_2x \leq \Sigma\theta x$ which completes the proof of (1).

Now suppose $x \in B$, $x \neq 0$, and $a \in At\mathfrak{A}$ such that $a \leq x$. Then $x \geq \Sigma\theta a = c_0a \cdot c_1a \cdot c_2a$, the smallest atom of $\mathfrak{B}$ that contains a. On the other hand, suppose x satisfies the condition. $\mathfrak{A}$ is atomic so $x = \Sigma\{a \in At\mathfrak{A} : a \leq x\} = \Sigma\{\Sigma\theta a : a \in At\mathfrak{A}, a \leq x\} \in B$ by (1) and the condition on x. ∎

Note that 1.6(iii) implies that an algebra $\mathfrak{B}$ which is minimal with respect to $\mathfrak{A} \in \mathbf{Adj}\,(\mathfrak{B})$ is unique up to isomorphism. The following is a corollary of the observation that θ^* is a bijection between E_{ij} and E'_{ij} for $i \neq j$. Namely, if $x \in E_{ij}$, $i \neq j$ and $y\theta x$, then $y = x$ because $x, y \leq d_{ij}$ and $x \cdot y = 0$ implies $c_ix \cdot c_iy = 0$; thus, $\theta x = \{x\}$ whenever $x \in E_{ij}$ for $i \neq j$.

Corollary 1.7. *Suppose* $\mathfrak{A} \in \mathbf{Adj}\,(\mathfrak{B})$. *Then* $\mathfrak{A}$ *is integral iff* $\mathfrak{B}$ *is integral.*

The notions of integral and adjunction as well as the results of this section can obviously be generalized to CA_n's for $n < \omega$.

2. Partial multi–valued loops

In this section we introduce a class of multi–valued systems that will be used to create CA_3's. Several special classes to be used later are described in 2.2.

Definition 2.1. *A partial multi–valued loop is a structure* $\mathfrak{M} = \langle M, \circ, E\rangle$ *where* $\emptyset \neq E \subseteq M$, $a \circ b \subseteq M$ *for all* $a, b \in M$, *and the following properties hold for all* $x, y \in M$:

(i) *there exist a unique* $e \in E$ *such that* $x \circ e = \{x\}$ *and a unique* $f \in E$ *such that* $f \circ x = \{x\}$. *We denote* $e = r(x)$ *and* $f = d(x)$.
(ii) $e \circ e = \{e\}$ *for all* $e \in E$,

(iii) $x \circ y \neq \emptyset$ *iff* $r(x) = d(y)$,
(iv) $x \in y \circ z$ *for some* $z \in M$ *iff* $d(x) = d(y)$,
(v) $x \in z \circ y$ *for some* $z \in M$ *iff* $r(x) = r(y)$,
(vi) *there exist a unique* $z \in M$ *and a unique* $w \in M$ *such that* $d(y) \in y \circ z$ *and* $r(y) \in wy$.

Intuitively, a partial multi–valued loop $\langle M, \circ, E\rangle$ can be thought of as a multi–pregroupoid in the sense of category theory. That is, there is a multi–valued composition $\circ$ on a set M of morphisms between objects of a set E. Each $x \in M$ has a unique domain and range object, $d(x)$ and $r(x)$ respectively. The objects are the idempotent maps. Conditions 2.1(iii)–(v) say that the composition is non–empty and "equations" can be solved precisely when the ranges and domains match properly. Property 2.1(vi) says that each map in M has a unique right and a unique left inverse. Note that the composition is not assumed to be associative.

When the value of a product is a singleton, i.e., $x \circ y = \{z\}$ as in 2.1(i) or 2.1(ii), we write $x \circ y = z$. A function on a partial multi–valued loop $\mathfrak{M}$ has a natural extension to subsets of M. The same symbol will denote both the function and its extension. For example, for $X, Y \subseteq M$, $X \circ Y = \bigcup\{x \circ y : x \in X, y \in Y\}$.

Examples 2.2. (i) A *multi–valued loop* (see [3]) is a partial multi–valued loop in which $E = \{e\}$. In this case, condition 2.1(i) implies that $x \circ e = x = e \circ x$ for all x which subsumes 2.1(ii). Conditions 2.1(iii)–(v) mean: given any two of x, y, z a third value exists such that $x \in y \circ z$. Condition 2.1(vi) stipulates unique right and left inverses.

(ii) A multi–valued loop (i.e., $E = \{e\}$) in which $|x \circ y| = 1$ for all $x, y \in M$ is an ordinary *loop*, i.e., $x \circ e = x = e \circ x$ for all x and each of the equations $x = y \circ z$, $y = x \circ z$, and $y = z \circ x$ have a unique solution for x given y and z.

(iii) A *polygroupoid* (see [5]) is a partial multi–valued algebra $\mathfrak{M} = \langle M, \circ, E, ^{-1}\rangle$ where $\emptyset \neq E \subseteq M$, $a \circ b \subseteq M$ for all $a, b \in M$ and $^{-1}$ is a unary operation on M such that

(1) $(x \circ y) \circ z = x \circ (y \circ z)$ for all x, y, z,
(2) $x \circ E = x = E \circ x$ for all x,
(3) the formulas $x \in y \circ z$, $y \in x \circ z^{-1}$, and $z \in y^{-1} \circ x$ are equivalent for all x, y, z.

Using properties (1)–(3) and Lemma 4.1 of [5] it is not hard to show that $\langle M, \circ, E\rangle$ is a partial multi–valued loop whenever $\langle M, \circ, E, ^{-1}\rangle$ is a polygroupoid. The atom structures of relation algebras were characterized by polygroupoids in [5].

(iv) A *polygroup* is a polygroupoid (see (iii) above) in which $E = \{e\}$ and $a \circ b \neq \emptyset$ for all $a, b \in M$. Many natural examples of polygroups are obtained from groups, geometry, and combinatorial schemes (cf., [6], [7]). The atom structures of integral relation algebras were characterized by polygroups in [6].

(v) A *groupoid*, i.e., a category in which every morphism is invertible, can be regarded as a partial multi–valued loop and, in fact, a polygroupoid in a natural way (cf., [10]). Namely, a groupoid can be defined as a system $\langle M, \circ, I\rangle$ where I is the set of objects; M is the disjoint union of $\{M_{ij} : i, j \in I\}$ where M_{ij} is the set of morphisms from i to j; $\circ$ is a partial binary operation where $x \circ y$ is defined for $x \in M_{ij}$ and $y \in M_{st}$ iff $j = s$, and, in this case, $x \circ y \in M_{it}$; $i \in M_{ii}$ is an identity (i.e., $i \circ x = x$ and $y \circ i = y$ for $x \in M_{ij}$ and $y \in M_{ki}$); $\circ$ is associative whenever it makes sense; and each $x \in M_{ij}$ has a unique inverse $y \in M_{ji}$ such that $i = x \circ y$ and $j = y \circ x$.

The following lemma summarizes some elementary consequences of the axioms for partial multi–valued loops.

Lemma 2.3. *If $\langle M, \circ, E\rangle$ is a partial multi–valued loop, the following hold for all $x, y, z \in M$:*

(i) *if $x \in E$, then $d(x) = r(x) = x$,*
(ii) *if $x, y \in E$ and $x \circ y \neq \emptyset$, then $x = y$,*
(iii) *if $x \in E$ and $x \circ y \neq \emptyset$, then $x \circ y = y$,*
(iv) *if $x \in E$ and $y \circ x \neq \emptyset$, then $y \circ x = y$,*
(v) *if $x \in y \circ z$, then $d(y) = d(x)$, $r(y) = d(z)$, and $r(z) = r(x)$,*
(vi) *if $x \in y \circ z$ and either $x, y \in E$ or $x, z \in E$ or $y, z \in E$, then $x = y = z \in E$.*

Proof. (i) By 2.1(ii) $x \circ x = x$ and, by 2.1(i), $r(x)$ is the unique $e \in E$ with $x \circ e = x$. Thus, $r(x) = x$. Similarly, $d(x) = y$.

(ii) If $x \circ y \neq 0$, then $x = r(x) = d(y) = y$ by 2.1(iii) and 2.3(i).

(iii) Suppose $x \circ y \neq \emptyset$ and $x \in E$. Then $x = r(x) = d(y)$ by 2.3(ii) and 2.1(iii). Thus, $x \circ y = d(y) \circ y = y$ by 2.1(i).

(iv) is similar to (iii) and (v) follows from 2.1(iii)–(v).

(vi) Suppose $x \in y \circ z$. If $y \in E$, 2.3(iii) implies that $x \in y \circ z = z$ so $x = z$ and hence $x \in y \circ x$. If either $x \in E$ or $z \in E$, 2.3(iv) implies that $x \in y \circ x = y$ from which $y = x (= z)$ follows. The case where $x, z \in E$ is similar. ∎

We conclude this section by associating a cylindric atom structure with a partial multi–valued loop. The construction is an obvious generalization of the construction given in Monk [12] to produce a cylindric atom structure of a non–representable CA_3 from a quasigroup. See 3.2.72 of [9] for another generalization.

Definition 2.4. *Suppose* $\mathfrak{M} = \langle M, \circ, E\rangle$ *is a partial multi–valued loop. Let* $\mathfrak{A}_{\mathfrak{M}} = \langle R, T_i, E_{ij}\rangle_{i,j<3}$ *where* $R = \{(x,y,z) \in M^3 : z \in x \circ y\}$ *and for* $\{i,j,k\} = \{0,1,2\}$, $T_i = \{(u,v) \in R^2 : u_i = v_i\}$, $E_{ii} = R$, *and,* $E_{ij} = T_k\{(e,e,e) : e \in E\}$.

Lemma 2.5. $\mathfrak{A}_{\mathfrak{M}}$ *is a cylindric atom structure whenever* $\mathfrak{M}$ *is a partial multi–valued loop.*

Proof. We need to verify conditions 1.3(ii), 1.3(iv), and 1.3(v) since, clearly, T_i is an equivalence relation and $E_{ii} = R$ for all $i < 3$.
1.3(ii): $T_i|T_j = T_j|Ti$.

Suppose $u, v \in R$. If $i = 0$ and $j = 1$,

$$\begin{aligned}(u,v) \in T_0|T_1 &\Leftrightarrow \exists w \in R(w_0 = u_0 \& w_1 = v_1) \Leftrightarrow r(u_0) = d(v_1) \text{ by 2.1(iii)} \\ &\Leftrightarrow d(u_1) = r(v_0) \text{ by 2.1(iii)} \\ &\Leftrightarrow (u,v) \in T_1|T_0\,.\end{aligned}$$

The cases $i = 0$, $j = 2$ and $i = 1$, $j = 2$ are similar using 2.1(iv), 2.1(v), and 2.3(v).
1.3(iv): $E_{ij} = T_k(E_{ik} \cap E_{kj})$ for $i, j \neq k$.

Assume first that $i = j$. We need to show that $R = E_{ii} = T_k(E_{ik}) = T_kT_l\{(e,e,e) : e \in E\}$ where $k \neq l \neq i \neq k$. Given $(x,y,z) \in R$, the following triples belong to R : $(x, r(x), x)$, $(x, w, d(x))$ for some w, $(d(y), y, y)$, $(w, y, r(y))$ for some w, $(z, r(z), z)$, and $(d(z), z, z)$. Using the appropriate triple, $(x,y,z)T_k|T_l(e,e,e)$ for some $e \in E$ whenever $k \neq l$. Now, suppose $i \neq j$. Obviously, $E_{ik} \cap E_{kj} \supseteq \{(e,e,e) : e \in E\}$. For the converse, suppose $u \in E_{ik} \cap E_{kj}$. Then $u_j \in E$ and $u_i \in E$. Thus, by 2.3(vi), $u_i = u_j = u_k$ so $u = (e,e,e)$ for some $e \in E$. Hence $E_{ik} \cap E_{kj} = \{(e,e,e) : e \in E\}$ from which 1.3(iv) follows.
1.3(v): $|T_iv \cap E_{ij}| = 1$ whenever $v \in R$ and $i \neq j$.

If $u \in T_iv \cap E_{ij}$, then $u_i = v_i$ and $u_k = e$ for some $e \in E$. For each triple (u_0, u_1, e), (e, u_1, u_2), and (u_0, e, u_2), if one of the u_i's, $u_i \neq e$, is known, the other is uniquely determined by 2.1(vi), 2.3(iii), and 2.3(iv). Thus, 1.3(v) holds and the lemma follows. ∎

3. Representability

The main results of this section give a representation of a CA_3 as an adjunction of the complex algebra of a partial multi–valued loop. In the case of an integral CA_3 the "coordinatizing system" $\mathfrak{M}_{\mathfrak{A}}$ in the representation is a multi–valued loop. The first result shows that a partial multi–valued loop can be associated with every complete and atomic CA_3 .

Suppose $\mathfrak{A}$ is a complete atomic CA_3 with atom structure $\langle At\mathfrak{A}, T_i,\ E_{ij}\rangle_{i,j<3}$. Define $\mathfrak{M}_{\mathfrak{A}} = \langle E_{12}, \circ, E\rangle$ where $E = E_{01} \cap E_{12}$ and, for $x, y \in E_{12}$,

$$x \circ y = E_{12} \cap T_2(T_0(d_{02} \cdot c_2 x) \cap T_1 y)$$

Proposition 3.1.

(i) *$\mathfrak{M}_{\mathfrak{A}}$ is a partial multi–valued loop if $\mathfrak{A}$ is a complete atomic CA_3.*

(ii) *If $\mathfrak{A}$ is integral, then $\mathfrak{M}_{\mathfrak{A}}$ is a multi–valued loop.*

Proof. (ii) follows easily from (i) because if $\mathfrak{A}$ is integral, then $E = \{d_{01} \cdot d_{12}\}$ by Lemma 1.1. To verify (i) we must check properties 2.1(i)–(vi). Observe that for $x, y, z \in E_{12}$:

(1) $$z \in x \circ y \Leftrightarrow z \leq d_{12} \cdot c_2(s_2^0 c_2 x \cdot c_1 y)\,.$$

where $s_j^i x = c_i(d_{ij} x)$ for $i \neq j$ (cf., 1.5.1 of [8]).

(2) for $x \in E_{12}$, $f = d_{02} \cdot c_0 x \in E$ and $f \circ x = x\,.$

Clearly, $f = d_{02} \cdot c_0 x \leq d_{02} d_{12}$ and f is an atom by 1.10.4(ii) of [8]. Thus, $f \in E$. Now, by (1),

$$z \in f \circ x \Leftrightarrow z \leq d_{12} \cdot c_2(c_0 f \cdot c_1 x) = d_{12} \cdot c_2(c_0 x \cdot c_1 x)\,.$$

But, $c_0 x \leq c_0 d_{12} = d_{12}$; so $c_0 x \cdot c_1 x = c_0 x \cdot d_{12} \cdot c_1 x = x \cdot c_0 x = x$. Therefore, $z \in f \circ x \Leftrightarrow z \leq d_{12} \cdot c_2 x = x$. Hence $f \circ x = x$ and (2) holds.

(3) for $x \in E_{12}$, $e = d_{12} \cdot c_1(d_{02} \cdot c_2 x) = d_{12} \cdot d_{02} \cdot c_1 c_2 x \in E$ and $x \circ e = x$.

An argument similar to the one for (2) shows $e \in E$. Then $x \circ e = x$ follows from

$$z \in x \circ e \Leftrightarrow z \leq d_{12} \cdot c_2(s_2^0 c_2 x \cdot d_{02} \cdot c_1 c_2 x) =$$
$$d_{12} \cdot c_2(c_2 x \cdot d_{02} \cdot c_1 c_2 x) = d_{12} \cdot c_2 x = x\,.$$

(4) for $x \in E_{12}$ and $e, e' \in E$, if either $x \circ e \neq \emptyset$ and $x \circ e' \neq \emptyset$ or $e \circ x \neq \emptyset$ and $e' \circ x \neq \emptyset$, then $e = e'$.

Suppose $x \circ e \neq \emptyset$. Then $0 \neq c_2(d_{02} \cdot c_2 x \cdot c_1 e) = c_2 x \cdot c_2 c_1 e = c_2(x \cdot c_2 c_1 e)$ using (1), 1.3.9 of [8], (C_3), and $c_1 e \leq d_{02}$. Thus, $x \leq c_2 c_1 e$ because $x \in At\mathfrak{A}$ and $x \cdot c_2 c_1 e \neq 0$. Also if $x \circ e' \neq \emptyset$, then $x \leq c_2 c_1 e \cdot c_2 c_1 e'$. It follows that $e = e'$ because if $e \neq e'$, then $e \cdot e' = 0$ and $e, e' \leq d_{12}$ which implies $c_1 e \cdot c_1 e' = 0$ and $c_1 e, c_1 e' \leq d_{02}$ which gives $c_2 c_1 e \cdot c_2 c_1 e' = 0$ contradicting $x \leq c_2 c_1 e \cdot c_2 c_1 e'$. The proof that $e \circ x \neq \emptyset$ and $e' \circ x \neq \emptyset$ implies $e = e'$ is similar.

The existence part of 2.1(i) follows from (2) and (3) while the uniqueness part follows from (4). Moreover, it follows from (2), (3), and (4) that

(5) for $x \in E_{12}$, $d(x) = d_{02} \cdot c_0 x$ and $r(x) = d_{12} \cdot d_{02} \cdot c_1 c_2 x$ and that if $x \circ e \neq \emptyset$, $f \circ x \neq \emptyset$ for $e, f \in E$, then $e = r(x)$ and $f = d(x)$.

Condition 2.1(ii) follows from (6) below.

(6) for $e \in E$, $e \circ e = e$.

Using (1), the fact $e \leq d_{02}$, $c_0 e \leq d_{12}$, and 1.3.9 of [8] we obtain

$$z \in e \circ e \Leftrightarrow z \leq d_{12} \cdot c_2(c_0 e \cdot c_1 e) = d_{12} \cdot c_2(c_0 e \cdot d_{12} \cdot c_1 e) = d_{12} \cdot c_2 e = e\,.$$

(7) for $x, y \in E_{12}$, $x \circ y \neq \emptyset \Leftrightarrow x \circ e = x$ and $e \circ y = y$ for some $e \in E$.

First note that for $x, y \in E_{12}$,

$$\begin{aligned} x \circ y \neq \emptyset &\Leftrightarrow s_2^0 c_2 x \cdot c_1 y \neq 0 \\ &\Leftrightarrow c_0 c_1(d_{02} \cdot c_2 x) \cdot c_0 c_1 y \neq 0 \\ &\Leftrightarrow c_1(d_{02} \cdot c_2 x) \cdot c_2 y \neq 0\,. \end{aligned} \tag{8}$$

Also, $c_1(d_{02} \cdot c_2 x) \cdot c_0 y \leq d_{02} \cdot d_{12}$ because $c_1 d_{02} = d_{02}$ and $c_0 y \leq c_0 d_{12} = d_{12}$. To prove $\Longrightarrow$, assume $x \circ y \neq \emptyset$ and let $e = c_1(d_{02} \cdot c_2 x) \cdot c_0 y$. By an argument similar to the one used in the proof of (2), $e \in At\mathfrak{A}$ and thus $e \in E$. Also,

$$\begin{aligned} z \in x \circ e &\Leftrightarrow z \leq d_{12} \cdot c_2(s_2^0 c_2 x \cdot d_{02} \cdot c_1 c_2 x \cdot c_0 c_1 y) \\ &\Leftrightarrow z \leq d_{12} \cdot c_2(c_2 x \cdot c_1 c_2 x \cdot c_0 c_1 y) \\ &\Leftrightarrow z \leq d_{12} \cdot c_2 x = x\,. \end{aligned}$$

Thus, $x \circ e = x$. Similarly, $e \circ y = y$ because $e \leq d_{02}$ implies $d_{02} \cdot c_2 e = e$ which, with $c_0 e \leq d_{12}$, gives

$$z \in e \circ y \Leftrightarrow z \leq d_{12} \cdot c_2(c_0 e \cdot c_1 y) \leq d_{12} \cdot c_1 y = y\,.$$

To prove $\Longleftarrow$, we assume $x \circ e \neq \emptyset$ and $e \circ y \neq \emptyset$. By (8) $x \circ e \neq \emptyset$ implies

$$0 \neq c_0c_1(d_{02} \cdot c_2x) \cdot c_0c_1e = c_0c_1(c_0c_1(d_{02} \cdot c_2x) \cdot e).$$

Therefore, $c_0c_1(d_{02} \cdot c_2x) \cdot e \neq 0$ which implies $e \leq c_0c_1(d_{02} \cdot c_2x)$. Similarly, $e \circ y \neq \emptyset$ implies $c_0(c_0c_1y \cdot e) \neq 0$ so $e \leq c_0c_1y$. Thus, $e \leq c_0c_1(d_{02} \cdot c_2x) \cdot c_0c_1y$ which implies $x \circ y \neq \emptyset$ by (8).

(9) for $x, y \in E_{12}$, $x \in y \circ z$ for some $z \in E_{12} \Leftrightarrow d(x) = d(y)$.

First, observe that for $x, y, z \in E_{12}$

(10) $x \in y \circ z \Leftrightarrow c_2x \leq c_2(s_2^0c_2y \cdot c_1z) \Leftrightarrow c_2x \cdot c_2(s_2^0c_2y \cdot c_1z \neq)0 \Leftrightarrow c_2x \cdot s_2^0c_2y \cdot c_1z \neq 0$

Now, assume $x \in y \circ z$ for some z. Then $c_2x \cdot s_2^0c_2y \cdot c_1z \neq 0$ for some z which implies $c_2x \cdot s_2^0c_2y \neq 0$. Hence, $c_0c_2x \cdot c_2s_2^0c_2y \neq 0$, $c_0c_2(c_0x \cdot c_2y) \neq 0$, and thus $c_0x \cdot c_2y \neq 0$. Since $c_0x \leq c_0d_{12} = d_{12}$ and $y \leq d_{12}$, $0 \neq c_0x \cdot c_2y = c_0x \cdot d_{12} \cdot c_2y = c_0x \cdot y$. Hence $y \leq c_0x$ and thus, $c_0x = c_0y$. Since the formula for $d(x)$ in (5) involves only c_0x, it follows that $d(x) = d(y)$. For the implication $\Leftarrow$, assume $d(x) = d(y)$. Then $c_0x = c_0y$ by (5) and so $0 \neq y \cdot c_0x \leq c_2y \cdot c_0x$. Thus,

$$c_0c_2(c_0x \cdot c_2y) = c_0c_2x \cdot c_2s_2^0c_2y = c_0c_2(c_2x \cdot s_2^0c_2y) \neq 0$$

and hence $c_2x \cdot s_2^0c_2y \neq 0$. Choose an atom $w \leq c_2x \cdot s_2^0c_2y$ and let $z = d_{12} \cdot c_1w$, an atom in E_{12}. Since $w \leq c_1z$ we have $c_2x \cdot s_2^0c_2y \cdot c_1z \neq 0$ and hence $x \in y \circ z$ by (10).

(11) for $x, y \in E_{12}$, $x \in z \circ y$ for some $z \in E_{12} \Leftrightarrow r(x) = r(y)$.

By an argument similar to the proof of (9) it can be shown that $x \in z \circ y$ for some z is equivalent to $c_2x \cdot c_1y \neq 0$ which, in turn, is equivalent to $c_1c_2x = c_1c_2y$; which is equivalent to $r(x) = r(y)$.

Conditions 2.1(iii)–(v) obviously follow from (7), (9), and (11). It remains to verify 2.1(vi). The existence portion follows from (9) and (11). The uniqueness is established below.

(12) For $y, z, z' \in E_{12}$ if $e \in z \circ y$ and $e \in z' \circ y$ where $e = r(y)$, then $z = z'$ and if $e \in y \circ z$ and $e \in y \circ z'$ where $e = d(y)$, then $z = z'$.

Suppose $e \in z \circ y$ where $e = r(y)$. Then $c_2e \leq d_{01}$ and, by (10), $c_2e \cdot s_2^0c_2z \cdot c_1y \neq 0$. Choose an atom $u \leq s_2^0c_2z \cdot c_1y \cdot d_{01}$. Then $c_1u \leq c_1y$, an atom in the BA of c_1–closed elements; so $c_1u = c_1y$. Thus, $u = d_{01} \cdot c_1u = d_{01} \cdot c_1y$. Also, $d_{02} \cdot c_0u \leq d_{02}s_2^0c_2z = d_{02} \cdot c_2z \in E_{02}$; so $d_{02} \cdot c_0u = d_{02} \cdot c_2z$.

Hence, $z = d_{12} \cdot c_2 z = d_{12} \cdot s_0^2 c_0 u$. If $e \in z' \circ y$, a similar argument shows that an atom $u' \leq s_2^0 c_2 z' \cdot c_1 y \cdot d_{01}$ has the form $u = d_{01} \cdot c_1 y = u$ and $z' = d_{12} \cdot s_0^2 c_0 u'$. Therefore, $z = z'$. The argument for the equations $e \in y \circ z$ and $e \in y \circ z'$ is similar: first choose an atom $u \leq s_2^0 c_2 y \cdot c_1 z \cdot d_{01}$ and show that $u = d_{01} \cdot s_2^0 c_2 y$ and $z = d_{12} \cdot c_1 u$ which depends only on y. ∎

By Proposition 3.1 each complete and atomic CA_3 $\mathfrak{A}$ gives rise to a partial multi-valued loop $\mathfrak{M}_{\mathfrak{A}}$ which, by Lemma 2.5, produces a cylindric atom structure. The next result characterizes this atom structure as exactly the cylindric atom structure associated with the minimal $\mathfrak{B}$ such that $\mathfrak{A} \in$ **Adj** $(\mathfrak{B})$.

Proposition 3.2. *For a complete atomic CA_3 $\mathfrak{A}$ let $\mathfrak{At}\mathfrak{A} = \langle At\mathfrak{A}, T_i, E_{ij}\rangle_{i,j,3}$ and let $\mathfrak{M} = \mathfrak{M}_{\mathfrak{A}} = \langle E_{12}, \circ, E\rangle$. Let $\mathfrak{A}_{\mathfrak{M}} = \langle R, T_i'', E_{ij}''\rangle_{i,j<3}$ denote the cylindric atom structure associated with $\mathfrak{M}$ in 2.4 and let $\mathfrak{At}\mathfrak{A}/\theta = \langle At\mathfrak{A}/\theta, T_i', E_{ij}'\rangle_{i,j<3}$ denotes the quotient cylindric atom structure where $\theta = T_0 \cap T_1 \cap T_2$.*

Then $\mathfrak{A}_{\mathfrak{M}} \cong \mathfrak{At}\mathfrak{A}/\theta$.

Proof. For $u \in R = \{(x, y, z) \in E_{12}^3 : z \in x \circ y\}$ define

$$\mathcal{S}(u) = T_0(d_{02} \cdot c_2 u_0) \cap T_1 u_1 \cap T_2 u_2 \,.$$

Note that $\mathcal{S}(u)$ is a non-empty subset of $At\mathfrak{A}$ because $u \in R$ (cf., (10) in 3.1). We will see below that $\mathcal{S}(u) = \theta x$ for some $x \in At\mathfrak{A}$. To show that $\mathcal{S}$ is the desired isomorphism of $\mathfrak{A}_{\mathfrak{M}}$ onto $\mathfrak{At}\mathfrak{A}/\theta$ we need the following properties of $\mathcal{S}$ for $u, v \in R$ and $x, y \in At\mathfrak{A}$:

(1) if $u \neq v$, then $\mathcal{S}(u) \cap \mathcal{S}(v) = \emptyset$,
(2) $x \in \mathcal{S}(u)$ where $u = (d_{12} \cdot s_0^2 c_0 x,\ d_{12} \cdot c_1 x,\ d_{12} \cdot c_2 x)$. Thus, $At\mathfrak{A} = \cup\{\mathcal{S}(u) : u \in R\}$,
(3) $\mathcal{S}((e, e, e)) = \{e\}$ for $e \in E = E_{01} \cap E_{12}$,
(4) $xT_i y$ iff $x \in \mathcal{S}(u)$, $y \in \mathcal{S}(v)$ with $u_i = v_i$,
(5) for $\{i, j, k\} = \{0, 1, 2\}$, $x \in E_{ij}$ iff $x \in \mathcal{S}(u)$ with $u_k \in E$.

By (4), if $x \in \mathcal{S}(u)$, $\theta x = (T_0 x) \cap (T_1 x) \cap (T_2 x) = \mathcal{S}(u)$. Hence, by (1) and (2), the map $u \mapsto \mathcal{S}(u)$ is a bijection of R onto $At\mathfrak{A}/\theta$. By (4), and appropriate definitions, $(\theta x)T_i'(\theta y)$ iff $xT_i y$ iff $u_i = v_i$ iff $uT_i''v$ where $x \in \mathcal{S}(u)$, $y \in \mathcal{S}(v)$. By (5), for $i \neq j$, $\theta x = \{x\} \in E_{ij}'$ iff $x \in E_{ij}$ iff $u_k \in E$ iff $u \in E_{ij}''$ where $x \in \mathcal{S}(u)$. Thus, $\mathfrak{A}_{\mathfrak{M}} \cong \mathfrak{At}\mathfrak{A}/\theta$.

It remains to verify properties (1)–(5). Property (1) follows from the fact that if $x, y \le d_{ij}$ and $x \circ y = 0$, then $c_i x \cdot c_i y = 0$. An easy calculation, using (1) in the proof of 3.1, shows that u given in (2) is in R:

$$d_{12} \cdot c_2(s_2^0 c_2(d_{12} \cdot s_0^2 c_0 x) \cdot c_1(d_{12} c_1 x)) = d_{12} \cdot c_2(s_2^0 c_0 x \cdot c_1 x) = d_{12} \cdot c_2(c_0 x \cdot c_1 x) \ge d_{12} \cdot c_2 x\,.$$

Then (2) follows because

$$\mathcal{S}(u) = T_0(d_{02} \cdot s_1^2 s_0^2 c_0 x) \cap T_1(d_{12} \cdot c_1 x) \cap T_2(d_{12} \cdot c_2 x) = T_0(x) \cap T_1(x) \cap T_2(x)$$

contains x.

For $e \le d_{01} \cdot d_{12} \le d_{02}$, $s_2^0 c_2 e = c_0 e \le d_{12}$. Thus, if $x \in \mathcal{S}((e,e,e))$, $x \le d_{12} \cdot c_1 e \cdot c_2 e = e$ from which (3) follows. To verify (4), suppose $xT_i y$ and choose u, v with $x \in \mathcal{S}(u)$, $y \in \mathcal{S}(v)$ as in (2). Note that $c_i x = c_i y$ implies $u_i = v_i$. Conversely, if $x \in \mathcal{S}(u)$, $y \in \mathcal{S}(v)$ and $u_i = v_i$ it is immediate that $xT_i u_i T_i y$ for $i = 1, 2$. For $i = 0$, $xT_0(d_{02} \cdot c_2 u_0) = (d_{02} c_2 v_0) T_0 y$; so (4) holds. For (5), suppose $i \ne j$. Then, by (C_6), $d_{ij} = c_k(d_{ik} d_{kj})$; so

$$\begin{aligned} x \in E_{ij} \quad &\text{iff } xT_k e \text{ for some } e \in E_{ik} \cap E_{kj} = E_{01} \cap E_{12} \\ &\text{iff } x \in \mathcal{S}(u) \text{ with } u_k \in E \end{aligned}$$

using (3) and (4). This completes the proof of (5) and the proof of Proposition 3.2. ∎

If $\mathfrak{M}$ is a partial multi–valued loop, let $\mathfrak{B}[\mathfrak{M}]$ denote $\mathfrak{Cm}\mathfrak{A}_{\mathfrak{M}}$ where $\mathfrak{A}_{\mathfrak{M}}$ is the cylindric atom structure associated with $\mathfrak{M}$ in 2.4. If K is a class of partial multi–valued loops, let $\mathfrak{B}[K]$ denote $\{\mathfrak{B}[\mathfrak{M}] : \mathfrak{M} \in K\}$. We refer to $\mathfrak{B}[\mathfrak{M}]$ as the *complex algebra* of $\mathfrak{M}$. The next result, which is a corollary to 3.2 and 1.6, characterizes these complex algebras.

Theorem 3.3. *For a complete atomic CA_3 $\mathfrak{A}$ the following are equivalent:*

(i) $\mathfrak{A} \cong \mathfrak{B}[\mathfrak{M}]$ *for some partial multi–valued loop* $\mathfrak{M}$,

(ii) $c_0 x \cdot c_1 x \cdot c_2 x = x$ *for every* $x \in At\mathfrak{A}$,

(iii) *for every* $x, y \in At\mathfrak{A}$, *if* $c_i x = c_i y$ *for all* $i < 3$, *then* $x = y$.

Proof. (i) $\Longrightarrow$ (ii). Suppose $u, v \in R$ (= the graph of $\circ$) and $u \in c_0\{v\} \cdot c_1\{v\} \cdot c_2\{v\}$. Then $u_i = v_i$ for all $i < 3$ so $u = v$, i.e., $c_0\{v\} \cdot c_1\{v\} \cdot c_2\{v\} = \{v\}$.

(ii) $\Longrightarrow$ (iii) is obvious. Now assume (iii) and choose a minimal $\mathfrak{B}$ such that $\mathfrak{A} \in \mathbf{Adj}(\mathfrak{B})$, cf. 1.6. By (iii), $\mathfrak{A} = \mathfrak{B}$, i.e., no atom of $\mathfrak{B}$ can be composed of more than a single atom of $\mathfrak{A}$. By Proposition 3.2, the cylindric atom structure of $\mathfrak{A}$ is isomorphic to $\mathfrak{A}_{\mathfrak{M}}$ where $\mathfrak{M} = \mathfrak{M}_{\mathfrak{A}}$. Hence, $\mathfrak{A} \cong \mathfrak{B}[\mathfrak{M}_{\mathfrak{A}}]$, so (i) holds. ∎

The main representation results now follow. In a sense 3.4(i) says the partial multi–valued loop $\mathfrak{M}_{\mathfrak{A}}$ coordinatizes $\mathfrak{A}$. The class of partial multi–valued loops (multi–valued loops) is denoted by *pMV–LOOP* (*MV–LOOP*, respectively).

Theorem 3.4. (i) *For every complete atomic* CA_3, $\mathfrak{A}$, $\mathfrak{A} \in$ **Adj** $(\mathfrak{B}[\mathfrak{M}_{\mathfrak{A}}])$.

(ii) *In (i),* $\mathfrak{A}$ *is integral iff* $\mathfrak{M}_{\mathfrak{A}}$ *is a multi–valued loop.*

(iii) $CA_3 =$ **SAdj** $(\mathfrak{B}[pMV\text{–}LOOP])$.

(iv) *The class of integral* CA_3*'s is* **SAdj** $(\mathfrak{B}[MV\text{–}LOOP])$.

Proof. (i) holds by 1.6 and 3.2. (ii) follows from 3.1(ii), 1.7, and the observation that $\mathfrak{B}[\mathfrak{M}]$ is integral whenever $\mathfrak{M}$ is a multi–valued loop by Lemma 1.1. Parts (iii) and (iv) follow from (i) and (ii) using Lemma 1.2. ∎

In the remainder of this section we characterize several natural classes of CA_3's in the spirit of Theorems 3.3 and 3.4.

Theorem 3.5. *Suppose* $\mathfrak{A}$ *is a complete atomic* CA_3. *Then:*

(i) $\mathfrak{A} \cong \mathfrak{B}[\mathfrak{M}]$ *for some multi–valued loop* $\mathfrak{M}$ *iff* $\mathfrak{A}$ *is integral and 3.3(ii) holds,*

(ii) $\mathfrak{A} \cong \mathfrak{B}[\mathfrak{M}]$ *for some loop* $\mathfrak{M}$ *iff* $\mathfrak{A}$ *is integral and*

$$(*) \qquad x = c_i x \cdot c_j x \quad \text{for every} \quad x \in At\mathfrak{A} \quad \text{and} \quad i < j < 3$$

(equivalently, $c_i x = c_i y$ *and* $c_j x = c_j y$ *imply* $x = y$ *whenever* $x, y \in At\mathfrak{A}$ *and* $i < j < 3$*).*

Proof. (i) holds by 3.3 and 3.4(ii).
(ii) $\Longrightarrow$: If $u, v \in R = \{x \in M^3 : x_2 \in x_0 \circ x_1\}$ such that $u \in c_i\{v\} \cdot c_j\{v\}$, then $u_i = v_i$ and $u_j = v_j$. Therefore, $u_k = v_k$ if $\mathfrak{M}$ is a loop. Thus, $u = v$ and $\{v\} = c_i\{v\} \cdot c_j\{v\}$, as desired. For $\Longleftarrow$, by 3.5(i) we may assume $\mathfrak{A} \cong \mathfrak{B}[\mathfrak{M}]$ where $\mathfrak{M}$ is a multi– valued loop. Suppose $c \in a \circ b$ and $c' \in a \circ b$ in $\mathfrak{M}$ and let $x = \{(a, b, c)\}$, $x' = \{(a, b, c')\} \in At\mathfrak{B}[\mathfrak{M}]$. Since $c_0 x = c_0 x'$ and $c_1 x = c_1 x'$, $x = x'$ and so $c = c'$, i.e., the operation $\circ$ on $\mathfrak{M}$ is single–valued. ∎

In view of 3.5(ii), to characterize the complex algebras of groups we only need to find a CA_3 identity that expresses the associative law in a multi–valued loop. The following equation was used in Monk [12].

$$(E_1) \qquad c_2(c_1 z \cdot s_2^0 c_2(c_0 x \cdot c_1 y)) = c_2(c_0 x \cdot s_2^1 c_2(c_1 z \cdot s_2^0 s_1^2 c_1 y))$$

Lemma 3.6. *For a partial multi-valued loop $\mathfrak{M}$, (E_1) holds in $\mathfrak{B}[\mathfrak{M}]$ iff $\mathfrak{M}$ is associative.*

Theorem 3.7. *Suppose $\mathfrak{A}$ is a complete atomic CA_3. Then:*

(i) *$\mathfrak{A} \cong \mathfrak{B}[\mathfrak{M}]$ for a group $\mathfrak{M}$ iff $\mathfrak{A}$ is integral and satisfies (E_1) and $(*)$ in 3.5(ii),*

(ii) *$\mathfrak{A} \cong \mathfrak{B}[\mathfrak{M}]$ for a polygroup $\mathfrak{M}$ iff $\mathfrak{A}$ is integral and satisfies 3.3(ii), (E_1), and the following identities:*

$$(E_2) \qquad s_2^1 s_0^2 s_1^0 c_1 x = s_0^1 s_2^0 s_1^2 c_1 x \quad \text{(cf., 2.6.44 of [8])}$$

$$(E_3) \qquad c_0 x \cdot c_1 y \cdot c_2 z \leq c_0 c_1 c_2 (d_{01} s_2^0 s_1^2 c_1 y \cdot c_1 (s_2^0 c_2 z \cdot s_0^2 c_0 x)),$$

$$(E_4) \qquad c_0 x \cdot c_1 y \cdot c_2 z \leq c_0 c_1 c_2 (d_{01} \cdot s_2^1 s_0^2 c_0 x \cdot c_0 (s_2^1 c_2 z \cdot s_1^2 c_1 y)).$$

Remark. The identity (E_2) guarantees that the unique left inverse of an element in 2.1(vi) is the same as its unique right inverse. (E_3) and (E_4) are consequences of the identity in 1.5.22 of [8] which holds in all cylindric set algebras. They are related to the implications in the definition of a polygroupoid 2.2(iii)(3).

Finally, we characterize the class of simple CA_3's. A partial multi-valued loop $\langle M, \circ, E \rangle$ is *connected* if for all $x, y \in E$ there exist z such that $x \circ z = z$ and $z \circ y = z$ (i.e., $d(z) = x$ and $r(z) = y$). This condition is obviously equivalent to: for all x, y in M there exist a z such that $x \circ z \neq \emptyset$ and $z \circ y \neq \emptyset$. We recall that a CA_3 is *simple* if it satisfies the property that $c_0 c_1 c_2 x = 1$ for all $x \neq 0$. The following result extends the characterization of simple relation algebras given in 4.5 and 4.6 of [5].

Theorem 3.8.

(i) *A complete atomic CA_3 $\mathfrak{A}$ is simple iff $\mathfrak{M}(\mathfrak{A})$ is connected.*

(ii) **SAdj** *$\{\mathfrak{B}[\mathfrak{M}] : \mathfrak{M}$ is connected$\}$ is exactly the class of all simple CA_3's.*

Proof. (i) $\Longleftarrow$: By 3.4(i), $\mathfrak{A} \in$ **Adj** $(\mathfrak{B}[\mathfrak{M}_\mathfrak{A}])$. Clearly $\mathfrak{A}$ is simple iff $\mathfrak{B}[\mathfrak{M}_\mathfrak{A}]$ is simple. Hence, it suffices to assume $\emptyset \neq X \subseteq R = \{(x, y, z) : z \in x \circ y\}$ and show that $c_0 c_1 c_2 X = R$. Suppose $(x, y, z) \in X$ and $(a, b, c) \in R$. Since $\mathfrak{M}_\mathfrak{A}$ is connected there exist u such that $d(u) = r(a)$ and $r(u) = r(z)$. By 2.1(v), $z \in v \circ u$ for some v, i.e., $(v, u, z) \in R$. By 2.1 $w \in a \circ u$ for some w. Thus,

$$(a, b, c) \in c_0\{(a, u, w)\} \subseteq c_0 c_1\{(v, u, z)\} \subseteq c_0 c_1 c_2\{(x, y, z)\} \subseteq c_0 c_1 c_2 X$$

as desired. The proof of $\Longrightarrow$ presents no difficulty.

(ii) follows immediately from 3.8(i) and the fact that perfect extensions preserve simplicity (cf., 2.7.17 of [8]). ∎

Remark 3.9. The partial multi–valued loop $\mathfrak{M}_{\mathfrak{A}}$ has $E_{12} = \{x \in At\mathfrak{A} : x \leq d_{12}\}$ as its universe. The elements in E_{12} correspond in a one–one way to atoms in $Nr_2\mathfrak{A}$ (= the BA of elements $x \in A$ such that $c_2x = x$). The correspondence sends $x \in E_{12}$ to $c_2x \in Nr_2\mathfrak{A}$ and its inverse sends $x \in AtNr_2\mathfrak{A}$ to $d_{12} \cdot x \in E_{12}$. Via this correspondence, the system $\mathfrak{M}_{\mathfrak{A}}$ is isomorphic to a structure defined on $AtNr_2\mathfrak{A}$, namely, to $\langle AtNr_2\mathfrak{A}, \circ, E'_{01}\rangle$ where $E'_{01} = \{x \in AtNr_2\mathfrak{A} : x \leq d_{01}\}$ and

$$x \circ y = \{z \in AtNr_2\mathfrak{A} : z \leq c_2(s_2^0x \cdot s_2^1y)\}\,.$$

In the definition of $x \circ y$ above, $c_2(s_2^0x \cdot s_2^1y) = y;x$ where ; is defined in 5.3.7 of [9]. Using $x;y$ in place of $y;x$ in the definition of $x \circ y$ above is not sufficient, however, to obtain the representation results of this section. Given a complete atomic integral CA_3 $\mathfrak{A}$ let $\mathfrak{N}_{\mathfrak{A}} = \langle AtNr_2\mathfrak{A}, \odot, E'_{01}\rangle$ where for $x, y \in AtNr_2\mathfrak{A}$,

$$x \odot y = \{z \in AtNr_2\mathfrak{A} : z \leq c_2(s_2^1x \cdot s_2^0y)\}\,.$$

It is not difficult to show that $\mathfrak{N}_{\mathfrak{A}}$ is a multi–valued loop anti–isomorphic to $\mathfrak{M}_{\mathfrak{A}}$. The complete atomic integral CA_3 $\mathfrak{A}_0 = \mathfrak{B}[\mathfrak{M}_0]$, constructed in Theorem 4.4 below, has the property that $\mathfrak{A}_0 \notin \mathbf{Adj}\,\mathfrak{B}[\mathfrak{N}_{\mathfrak{A}_0}]$.

4. Quasigroup representability

In this section we provide several applications and enhancements of the representation results in the previous section. In particular we settle Problems 1 and 2 in Monk [12] concerning the representation of CA_3's by quasigroups and groups.

We first describe the procedure Monk used to associate a CA_3 with a quasigroup $\langle Q, \circ\rangle$ together with a fixed triple $q \in Q^3$ such that $q_0 \circ q_1 = q_2$. Let $\mathfrak{A}_{Qq} = \langle R, T_i, E_{ij}\rangle_{i,j<3}$ where $R = \{(x,y,z) \in Q^3 : x \circ y = z\}$, $T_i = \{(u,v) \in R^2 : u_i = v_i\}$, $E_{ii} = R$, and $E_{ij} = \{u \in R : u_k = q_k\}$ whenever $\{i,j,k\} = 3$. $\mathfrak{A}_{Qq}$ is a cylindric atom structure whose complex algebra $\mathfrak{Cm}\mathfrak{A}_{Qq}$ is called a $Q, q - CA_3$. A CA_3 is called *quasigroup representable* if it is embeddable in some $Q, q - CA_3$.

A quasigroup together with a distinquished element in the graph of $\circ$ is technically not one of the structures described in Definition 2.1. However, this system is isotopic to such a structure. Below we see that Q, q–CA_3's are exactly those that are embeddable in the complex algebras of loops.

Two binary systems $\langle G, \cdot \rangle$ and $\langle H, \circ \rangle$ are *isotopic* if there exist a triple (α, β, γ) of one–one maps of G onto H such that

$$\alpha(x) \circ \beta(x) = \gamma(x \cdot y)$$

for all $x, y \in G$. Every quasigroup is isotopic to a loop and, as mentioned in Bruck [1], the isotopism (α, β, γ) can be chosen so that $\alpha(q_0) = \beta(q_1) = \gamma(q_2) = e$ where e is the identity element of the loop and (q_0, q_1, q_2) is a fixed triple such that $q_0 \cdot q_1 = q_2$. This fact together with 4.1(i) yields 4.1(ii) below.

Proposition 4.1.

(i) *If (α, β, γ) is an isotopism between quasigroups $(Q, \cdot)$ and $(H, \circ)$ and $q \in Q^3$, $h \in H^3$ such that $\alpha(q_0) = h_0$, $\beta(q_1) = h_1$ and $\gamma(q_2) = h_2$, then $\mathfrak{Cm}\mathfrak{A}_{Qq} \cong \mathfrak{Cm}\mathfrak{A}_{Hh}$ where the isomorphism is induced by $f((x, y, z)) = (\alpha x, \beta y, \gamma z)$.*

(ii) *The class of Q, q–CA_3's is precisely $\mathfrak{B}[LOOP]$ where $LOOP$ denotes the class of all loops.*

(iii) *$\mathbf{S}\mathfrak{B}[LOOP]$ is the class of quasigroup representable CA_3's.*

The notion of isotopism and 4.1(i) extend to a quasigroup analogue of a partial multi–valued loop. In general, a "partial multi–valued quasigroup" is isotopic to a partial multi–valued loop and isotopic structures produce isomorphic CA_3's. Thus, 4.1 and its generalization tell us that, as far as representing CA_3 's are concerned, there is no need to consider quasigroup type systems; loops will do.

The next goal (Theorem 4.4) is to show there are integral CA_3 's that are not quasigroup representable. The argument will involve showing that a certain sentence holds in the class **SAdj** $\mathfrak{B}[LOOP]$.

A variable v in a formula φ is *cylindric dependent* if there is an i (depending on v) such that each occurrence of v, not part of a quantifier, is in a subterm $c_i v$. Note that every variable in the equations (E_1)–(E_4) is cylindric dependent, but x in the formula $c_0 x \cdot c_1 x \leq d_{01}$ is not.

A basic Horn formula $\theta_0 \vee \cdots \vee \theta_n$ is a *strict basic Horn formula* if exactly one θ_i is atomic and, of course, the others are negated atomic formulas. A *strict Horn sentence* is built up from strict basic Horn sentences using $\wedge, \exists$, and $\forall$.

For a complete atomic CA_3 $\mathfrak{A}$ and $a \in A$, 1.6(iii) implies that, if $a \leq d_{ij}$ for $i \neq j$, then a belongs to the minimal subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{A} \in \mathbf{Adj}(\mathfrak{B})$. This observation and the fact, that $c_i(d_{ij} \cdot c_i x) = c_i x$ whenever $i \neq j$, are used in the proof below.

Proposition 4.2. *Suppose $\mathfrak{A}$ is a complete atomic CA_3, $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ such that $\mathfrak{A} \in \mathbf{Adj}(\mathfrak{B})$, and φ is a universal strict Horn sentence such that every variable v which occurs in φ is either cylindric dependent or v has an occurrence in a negated atomic clause $\neg\psi$ of φ such that ψ implies $v \leq d_{ij}$ for some $i \neq j$. Then*

φ holds in $\mathfrak{A}$ iff φ holds in $\mathfrak{B}$.

Proof. For the non–trivial implication assume that φ has the form

$$\forall v_0 \ldots \forall v_m(\neg\theta_0 \vee \cdots \vee \neg\theta_{n_1} \vee \theta_n)$$

where θ_i is atomic for $i < n+1$, φ holds in $\mathfrak{B}$, and $x \in A^\omega$ is a sequence that satisfies $\theta_0 \wedge \cdots \wedge \theta_{n-1}$ in $\mathfrak{A}$. Define a sequence x' as follows:

$$x'_j = \begin{cases} d_{ki} \cdot c_k x_j & \text{if } v_j \text{ is cylindric dependent on } c_k \text{and } i \text{ is the least} \\ & l < 3, l \neq k \\ 0 & \text{if } v_j \text{ does not occur in } \varphi \\ x_j & \text{otherwise} \end{cases}$$

Then $x' \in B^\omega$ and x' satisfies $\theta_0 \wedge \cdots \wedge \theta_{n-1}$ in $\mathfrak{B}$. Since φ holds in $\mathfrak{B}$, it follows that x' satisfies θ_n in $\mathfrak{B}$ and hence in $\mathfrak{A}$. Because the values of x', different from the values of x, are assigned to cylindric dependent variables, x satisfies θ_n in $\mathfrak{A}$. Hence, φ holds in $\mathfrak{A}$. ∎

Lemma 4.3. *Suppose $\mathfrak{B}$ is a complete atomic integral CA_3, $x \neq 0$, and $c_i x \cdot c_j x \leq d_{ij}$ for some $i \neq j$. Then x is an atom.*

Proof. Suppose $a, b \in At\mathfrak{B}$, $a, b \leq x \leq c_i x \cdot c_j x \leq d_{ij}$. Choose an atom $y \leq c_i a \cdot c_j b$. [This is possible since $\mathfrak{B}$ is integral.] Then $y \leq c_i x \cdot c_j x \leq d_{ij}$ and $c_i y = c_i a$ so $y = d_{ij} \cdot c_i y = d_{ij} \cdot c_i a = a$. Similarly, $y = b$; so $a = b$ and x is an atom $(= c_i x \cdot c_j x)$. ∎

The following result gives a negative answer to Problem 1 posed in Monk [12].

Theorem 4.4. *Let $\mathfrak{M}_0 = \langle\{0,1,2\}, \circ, \{0\}\rangle$ where $\circ$ is given by the table*

$\circ$	0	1	2
0	0	1	2
1	1	2	0,2
2	2	0	1

*Then $\mathfrak{B}[\mathfrak{M}_0] \in [MV\text{–}LOOP]$, but $\mathfrak{B}[\mathfrak{M}_0] \notin$ **SAdj** $\mathfrak{B}[LOOP]$. Thus, $\mathfrak{B}[MV\text{–}LOOP] \not\subseteq$ **SAdj** $\mathfrak{B}[LOOP]$.*

Proof. $\mathfrak{M}_0$ is clearly a multi–valued loop. Consider the sentence ψ:

$$\forall x, y, z(c_0x \cdot c_2x \leq d_{02} \wedge c_1y \cdot c_1z = 0 \rightarrow c_2(c_1y \cdot c_0x) \cdot c_2(c_1z \cdot c_0x) = 0)\,.$$

ψ fails in $\mathfrak{B}[\mathfrak{M}_0]$ under the assignment $x \rightarrow a = \{(1,0,1)\}$, $y \rightarrow b = \{(1,2,2)\}$, and $z \rightarrow c = \{(1,1,2)\}$.

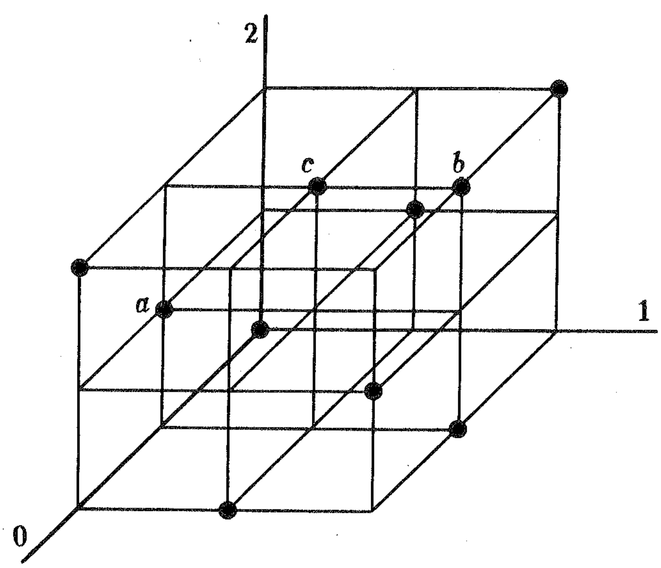

Figure 1

The theorem follows from the fact that ψ holds in **SAdj** $\mathfrak{B}[LOOP]$. Since ψ is a universal strict Horn sentence that satisfies the hypothesis of Proposition 4.2, it suffices to show that ψ holds in $\mathfrak{B}[\mathfrak{M}]$ where $\mathfrak{M}$ is a loop.

Suppose x, y, z satisfy the hypothesis of ψ and $c_2(c_1y \cdot c_0x) \cdot c_2(c_1z \cdot c_0x) \neq 0$. Then $x \neq 0$ so, by Lemma 4.3, $x = \{(a, e, a)\}$ for some $a \in M$. Choose $u \in R$ (= the graph of $\circ$) such that $u \in c_2(c_1y \cdot c_0x) \cdot c_2(c_1z \cdot c_0x)$. Then there exist $v \in c_1y \cdot c_0x$ and $w \in c_1z \cdot c_0x$ such that $u_2 = v_2 = w_2$. Since $v \in c_0x$ and $w \in c_0x$, $v_0 = a = w_0$. From $v_2 = w_2$ and $v_0 = w_0$ we obtain $w = v \in c_1y \cdot c_1z$ because $\mathfrak{M}$ is a loop. This contradicts the hypothesis $c_1y \cdot c_1z = 0$. Hence ψ holds in $\mathfrak{B}[\mathfrak{M}]$ as desired. ∎

The 10 atom CA_3 $\mathfrak{B}[\mathfrak{M}_0]$ constructed in Theorem 4.4 has minimal size among integral CA_3's that are not quasigroup representable. Up to isomorphism there are exactly four 10 atom CA_3's that are not quasigroup representable. Three of these, like $\mathfrak{B}[\mathfrak{M}_0]$ do not belong to $\mathbf{SAdj}\,\mathfrak{B}[LOOP]$; the fourth is the algebra $\mathfrak{A}$ given in 4.6 below. The following proposition is the key step to show that every integral CA_3 with 9 or fewer atoms is in $\mathbf{S}\mathfrak{B}[GROUP]$.

Proposition 4.5. *Suppose $\mathfrak{J}$ is the polygroup $\langle\{0,1\}, \circ, 0\rangle$ with $\circ$ given by the table*

$\circ$	0	1
0	0	1
1	1	0,1

Then $\mathbf{SAdj}\,\mathfrak{B}[\mathfrak{J}] \subseteq \mathbf{S}\mathfrak{B}[GROUP]$.

The idea of the representation is to choose, for each $\mathfrak{A} \in \mathbf{Adj}\,\mathfrak{B}[\mathfrak{J}]$, a sufficiently large field F such that the element $a = \{(x, y, z) \in F^3 : z = x + y$ and $x, y, z \neq 0\}$ can be decomposed into enough subsets, each cylindrically equivalent to a, so that the subalgebra of $\mathfrak{B}[\langle F, +, 0\rangle]$ generated by these subsets is isomorphic to $\mathfrak{A}$.

The next example will show that the **Adj** operation cannot be eliminated from Theorem 3.4. A sentence involving the notion of a "small" element in $\mathfrak{A}$ will be used. The notion of "small" used here should not be confused with the notion in 3.1.56 of [9]. Intuitively, an atom $x \leq d_{ij}$ $(i \neq j)$ is small if $c_ix \cdot c_jx \leq d_{ij}$. An arbitrary element x in $\mathfrak{A}$ is small if its "coordinates" (cf., 3.2(2)) in the partial multi–valued loop $\mathfrak{M}_{\mathfrak{A}}$ are all small atoms. Formally, we let *small*(x) abbreviate the formula:

$$s_0^2c_0x \cdot s_2^1s_0^2c_0x \leq d_{12} \quad\wedge\quad c_1x \cdot s_1^2c_1x \leq d_{12} \quad\wedge\quad c_2x \cdot s_2^1c_2x \leq d_{12}\,.$$

Theorem 4.6. $\mathbf{Adj}\,\mathfrak{B}[GROUP] \not\subseteq \mathbf{S}\mathfrak{B}[MV\text{–}LOOP]$.

Proof. Let $\mathfrak{A} \in \mathbf{Adj}\,\mathfrak{B}[\mathbb{Z}_3]$ be obtained by adjoining an atom c' to act like $c = \{(1, 1, 2)\}$, see, e.g., 3.2.69 of [9]. Clearly, $\mathfrak{A} \in \mathbf{Adj}\,\mathfrak{B}[GROUP]$

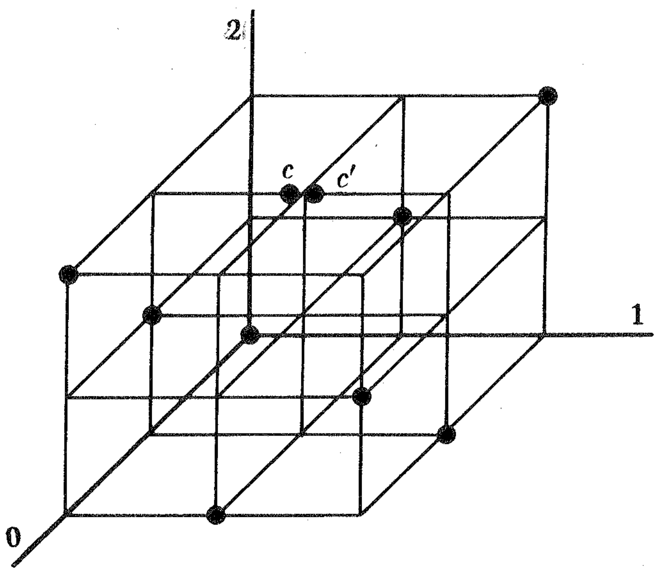

Figure 2

It remains to show that $\mathfrak{A} \notin \mathbf{S}\mathfrak{B}[MV\text{–}LOOP]$. Consider the sentence

$$\varphi: \qquad \forall x(x \neq 0 \wedge \text{small}(x) \to x \text{ is an atom}).$$

It is easy to check that φ fails in $\mathfrak{A}$ under the assignment $x \to c+c'$. Since φ is equivalent to a universal sentence, to prove the theorem, it suffices to show that φ holds in $\mathfrak{B}[\mathfrak{M}]$ where $\mathfrak{M}$ is a multi–valued loop. Suppose x satisfies the hypothesis of φ and let $a_0 = d_{12} \cdot s_0^2 c_0 x$, $a_1 = d_{12} \cdot c_1 x$, and $a_2 = d_{12} \cdot c_1 x$ (the "coordinates of x"). Then $0 \neq a_i \leq d_{12}$ for each i. Since x is small, Lemma 4.3 implies that each a_i is an atom, say $a_i = \{(e, u_i, u_i)\}$ for each $i < 3$. Now, suppose $v \in x$. Then $c_1 a_1 = c_1 x$ and $c_2 a_2 = c_2 x$ implies $v_1 = u_1$ and $v_2 = u_2$. Also, $c_2 a_0 = s_0^2 c_0 x$ implies $\{(u_0, e, u_0)\} = d_{02} \cdot c_2 a_0 = c_0 x$ which gives $v_0 = u_0$. Therefore, $x = \{(u_0, u_1, u_2)\}$ is an atom. Thus, φ holds in $\mathbf{S}\mathfrak{B}[MV\text{–}LOOP]$. Which completes the proof. ∎

The sentence that says

$$\forall x \forall y (c_0 x = c_0 y \wedge c_1 x = c_1 y \wedge c_2 x = c_2 y \wedge \text{ small}(x) \to x = y)$$

also works in the proof of Theorem 4.6. Does the algebra $\mathfrak{A}$ in 4.6 show $\mathbf{Adj}\,\mathfrak{B}[GROUP] \not\subseteq \mathbf{S}\mathfrak{B}[pMV\text{-}LOOP]$?

We now briefly consider the class RCA_3. Since $\mathbf{S}\mathfrak{B}[GROUP] \subseteq RCA_3$, Proposition 4.5 suggests it may be possible for every adjunction of an RCA_3 to be in RCA_3. We see below (Theorem 4.8) that this is not always the case. First, a technical lemma.

Lemma 4.7. *Suppose $\mathfrak{A}$ is a complete atomic integral RCA_3 isomorphic to a cylindric set algebra with base U and $|E_{ij}| < \omega$ for some $i \neq j$ such that*

(1) for every $x \in E_{ij}$, $c_i x \cdot c_j x \leq d_{ij}$.
Then $|U| = |E_{ij}|$.

Proof. Suppose F embeds $\mathfrak{A}$ into the full set algebra with base U. For i, j that satisfy the hypothesis define a relation $f_x \subseteq U \times U$ for each $x \in E_{ij}$ by

$$F(x) = \{u \in U^3 : u_i = u_j \quad \text{and} \quad (u_k, u_i) \in f_x\}$$

where $\{i, j, k\} = 3$. Then
(2) f_x is a function for each $x \in E_{ij}$.

For suppose $u, v \in F(x)$ and $u_k = v_k$. Let $w \in U^3$ with $w_j = u_j$, $w_i = v_i$ and $w_k = u_k = v_k$. Then, by (1), $w \in c_i\{u\} \cdot c_j\{v\} \leq D_{ij}$; so $w_i = w_j$ which gives $u = v$.

Since $\mathfrak{A}$ is integral
(3) domain$(f_x) = U =$range(f_x) for all $x \in E_{ij}$.

Since F is an embedding,
(4) $x \neq y$ implies $f_x \cap f_y = \emptyset$.

For $a, b \in U$ choose $v \in D_{ij} = \Sigma\{F(x) : x \in E_{ij}\}$ with $v_k = a$, and $v_i = v_j = b$. Because $|E_{ij}| < \omega$, $v \in F(x)$ for some x. This gives
(5) for all $a \in U$, $\cup\{f_x(a) : x \in E_{ij}\} = U$.

Now, fix $a \in U$. By (2)–(5), the map $x \mapsto f_x(a)$ gives a bijection of E_{ij} onto U. Hence $|U| = |E_{ij}|$. ∎

Theorem 4.8. *Suppose $\mathfrak{A}$ is a complete atomic integral RCA_3 for which $\mathfrak{M}_{\mathfrak{A}}$ is a finite loop. Then $\mathfrak{M}_{\mathfrak{A}}$ is a group and $\mathfrak{A} \cong \mathfrak{B}[\mathfrak{M}_{\mathfrak{A}}]$.*

Proof. If $\mathfrak{A} \in RCA_3$, then $\mathfrak{B}[\mathfrak{M}_{\mathfrak{A}}] \in RCA_3$. Monk has shown (Cor. 2.3 in [12] and also 3.2.73 of [9]) that if the complex algebra of a loop is in RCA_3 , then the loop is a group. Hence, we may assume $\mathfrak{M}_{\mathfrak{A}}$ is a group

G with n elements and $\mathfrak{A} \in \mathbf{Adj}\,\mathfrak{B}[G]$. Then $E_{ij}^{\mathfrak{A}} = E_{ij}^{\mathfrak{B}[G]}$ has n elements and $c_1x \cdot c_2x \leq d_{12}$ for all $x \in E_{ij}$. Suppose F is an isomorphism of $\mathfrak{A}$ onto a set algebra with base U. By Lemma 4.7, $|U| = n$. Since $c_0c_1F(x) = U^3$ for each $x \in At\mathfrak{A}$, $|F(x)| \geq n$. Also, $|At\mathfrak{B}[G]| = n^2$; so if $\mathfrak{A}$ is a proper adjunction of $\mathfrak{B}[G]$, $|At\mathfrak{A}| \geq n^2 + 1$. But then

$$|U^3| = \sum_{x \in At\mathfrak{A}} |F(x)| \geq (n^2 + 1)n > n^3 .$$

Thus, $\mathfrak{A}$ is not a proper adjunction of $\in [G]$, i.e., $\mathfrak{A} \cong \in [G]$. ∎

As a corollary we note that Theorem 4.8 implies that the CA_3 $\mathfrak{A}$ in Theorem 4.6 is not in RCA_3. Is it true that, for every partial multi–valued loop $\mathfrak{M}$, if $\mathfrak{B}[\mathfrak{M}]$ has some proper adjunction that is in RCA_3, then *every* proper adjunction of $\mathfrak{B}[\mathfrak{M}]$ is in RCA_3?

A representation of a complete CA_3 as a set algebra is called *completely additive* if the isomorphism preserves arbitrary sums. Our final discussion, concerning Problem 2 in Monk [12], will use results about polygroups and chromatic polygroups from Comer [6], [7].

A polygroup $\mathfrak{M}$ determines an integral relation algebra

$$\mathfrak{A}[\mathfrak{M}] = \langle SbM, \cup, \cap, \sim, \emptyset, M, \circ, ^{-1}, \{e\} \rangle$$

where $X \circ Y = \cup\{a \circ b : a \in X, b \in Y\}$ and $X^{-1} = \{a^{-1} : a \in X\}$ for all $X, Y \subseteq M$. The functor $\mathfrak{M} \mapsto \mathfrak{A}[\mathfrak{A}]$ gives a dual equivalence between the category of polygroups and the category of complete atomic integral RA's (cf., 3.3 of [6]) while the functor $\mathfrak{M} \mapsto \mathfrak{B}[\mathfrak{M}]$ gives a dual equivalence between polygroups and a full subcategory of the category of complete atomic integral CA_3's. The two functors $\mathfrak{A}[\mathfrak{M}]$ and $\mathfrak{B}[\mathfrak{M}]$ are closely related. In fact $\mathfrak{A}[\mathfrak{M}]$ can be intrinsically related to $\mathfrak{B}[\mathfrak{M}]$ by a correspondence similar to the construction given in 5.3.7 of [9] (cf., Remark 3.9). This paper makes no use of this correspondence. The following extends slightly Proposition 2.4 of [7].

Proposition 4.9.

(i) *For a multi–valued loop $\mathfrak{M}$, $\mathfrak{B}[\mathfrak{M}]$ has a completely additive representation iff $\mathfrak{M}$ is a chromatic polygroup.*

(ii) *If a complete atomic integral CA_3 $\mathfrak{A}$ has a completely additive representation, then $\mathfrak{M}_{\mathfrak{A}}$ is a chromatic polygroup.*

Proof. (i) Equations (E_1)–(E_4), used in Theorem 3.7, are among those known to hold in RCA_3's. Hence, if $\mathfrak{B}[\mathfrak{M}]$ is in RCA_3, $\mathfrak{M}$ is a polygroup. (i) now follows from 2.4 of [7].

(ii) follows immediately from 3.4, the fact $\mathfrak{B}[\mathfrak{M}_{\mathfrak{A}}]$ is embeddable in $\mathfrak{A}$, and part (i). ∎

The following gives a strong negative answer to Problem 2 in Monk [12].

Theorem 4.10. *There exist integral RCA_3's not in $\mathbf{S}\mathfrak{B}[GROUP]$. Moreover, the class $\mathbf{S}\mathfrak{B}[GROUP]$ is not finitely axiomatizable over the class of integral RCA_3's.*

Proof. The examples of integral RRA's not in $\mathbf{S}\mathfrak{A}[GROUP]$, constructed in McKenzie [11], have the form $\mathfrak{A}[\mathfrak{M}]$ where $\mathfrak{M}$ is a chromatic polygroup not in $Q_s(GROUP)$. (See [6].) For such an $\mathfrak{M}$, $\mathfrak{B}[\mathfrak{M}]$ is an integral RCA_3 by 4.9(i) and $\mathfrak{B}[\mathfrak{M}] \notin \mathbf{S}\mathfrak{B}[GROUP]$ by properties of the functor $\mathfrak{B}[-]$. By 2.2 and 4.2 of [7] there exist finite integral RCA_3's $\mathfrak{B}[\mathfrak{M}_i] \notin \mathbf{S}\mathfrak{B}[GROUP]$, for $i \in \omega$, such that an ultraproduct $\prod_F \mathfrak{B}[\mathfrak{M}_i]$ is in $\mathbf{S}\mathfrak{B}[GROUP]$. Thus, $\mathbf{S}\mathfrak{B}[GROUP]$ is not finitely axiomatizable. ∎

The non-finite axiomatizability proof above is a CA_3 version of the result in McKenzie [11] concerning group representable RA's.

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