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Perfect extensions of regular double Stone algebras

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Dedicated to the memory of Alan Day

Abstract. In 1951 Jónsson and Tarski showed that every Boolean algebra with operators could be embedded in a perfect (or canonical) extension. We obtain a similar result for regular double Stone algebras with operators. As a corollary we obtain another proof that every regular double Stone algebra can be represented as an algebra of rough subsets of an approximation space.

1. Introduction

A *double Stone algebra* $\mathbf{A} = \langle A, +, \cdot, *, ^+, 0, 1\rangle$ is an algebra of type $\langle 2, 2, 1, 1, 0, 0\rangle$ such that

(i) $\langle A, +, \cdot, 0, 1\rangle$ is a bounded distributive lattice;
(ii) $*$ is a pseudocomplement (i.e., $a \cdot x = 0$ iff $x \leq a^*$) satisfying the Stone identity $x^* + x^{**} = 1$;
(iii) $^+$ is a dual pseudocomplement (i.e., $a + x = 1$ iff $x \geq a^+$) satisfying the dual Stone identity $x^+ \cdot x^{++} = 0$.

A double Stone algebra $\mathbf{A}$ is called *regular* if $x^* = y^*$ and $x^+ = y^+$ imply $x = y$.
For standard facts about double Stone algebras we refer the reader to Grätzer [5] or Balbes and Dwinger [1]. For a double Stone algebra $\mathbf{A}$ we let $C(\mathbf{A}) = \{a^* : a \in A\}$ denote the *center* of A. $C(\mathbf{A})$ is a Boolean algebra that is a subalgebra of $\mathbf{A}$ and can be described in various ways:

$$C(\mathbf{A}) = \{a \in A : a = a^{**}\} = \{a^+ : a \in A\}$$
$$= \{a \in A : a^* = a^+\} = \{a \in A : a = a^{++}\}.$$

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Also, let $D(\mathbf{A}) = \{a \in A : a^* = 0\}$, the filter of dense elements of $\mathbf{A}$, and $F = D(\mathbf{A})^{++} = \{a^{++} : a \in D(\mathbf{A})\}$ be the corresponding filter on $C(\mathbf{A})$. We denote the complement of an element x in a Boolean algebra by x' to distinguish it from the two pseudocomplements.

In the paper [8] by T. Katriňák it is shown that every regular double Stone algebra $\mathbf{A}$ is uniquely determined by the pair $(C(\mathbf{A}), F)$. In fact, if $\mathbf{C}$ is a Boolean algebra and F is a filter on $\mathbf{C}$, then

$$A = \{(a, b) \in C^2 : a \geq b, b + a' \in F\}$$

has the structure of a regular double Stone algebra denoted by $\mathbf{A} = (\mathbf{C}, F)$. To see this, regard $\mathbf{A}$ as a bounded sublattice of $\mathbf{C}^2$ and define the pseudocomplements for $x = (a, b) \in A$ by

$$x^* = (a', a') \quad \text{and} \quad x^+ = (b', b').$$

Moreover, every $\mathbf{A}$ can be represented in this way by the map that sends $x \in A$ to the pair (x^{**}, x^{++}). This representation is used throughout the paper.

Now let $\mathbf{S}_2$ $\mathbf{S}_3$, and $\mathbf{S}_4$ denote the two, three and four element chains considered as double Stone algebras. Katriňák [9] showed that these are the only non-trivial subdirectly irreducible double Stone algebras. $\mathbf{S}_3$ generates the subvariety of regular algebras which is the variety of double Stone algebras that covers the variety of Boolean algebras (cf., [14]).

In 1951 Jónsson and Tarski showed that every Boolean algebra (BA) could be embedded in a complete atomic BA in such a way that additive operations could be extended and certain identities between the operations were preserved. In this paper we extend part of the Jónsson–Tarski theory of perfect (canonical) extensions to the variety of regular double Stone algebras. The basic idea for constructing the perfect extension of a Boolean algebra $\mathbf{A}$ is to regard $\mathbf{A}$ as the algebra of all clopen subsets of a Boolean space X and to let the perfect extension of $\mathbf{A}$ be the power set $\mathbf{P}(X)$. Observe that this construction can be viewed in a slightly different way. Namely, the algebra of all clopen subset of X can be viewed as the algebra of continuous sections of the trivial sheaf over X and $\mathbf{P}(X)$ as the set of all functions from X into $\mathbf{2}$. The idea for obtaining the perfect extension $\mathbf{A}^\sigma$ of a regular double Stone algebra $\mathbf{A}$ is to represent $\mathbf{A}$ as the algebra of continuous sections of a sheaf (X, S) over a Boolean space X (cf., [2]) and take $\mathbf{A}^\sigma$ as the direct product of the resulting stalks $\prod_{x \in X} \mathbf{S}_x$.

Jónsson and Tarski algebraically characterized the relationship between a Boolean algebra $\mathbf{A}$ and its perfect extension $\mathbf{A}^\sigma$. We do the same for regular double Stone algebras, but will use the representation of Katriňák [8] to avoid references to sheaves.

The relationship between **A** and its perfect extension $\mathbf{A}^\sigma$ will be described in Section 2 and the extension of operators will be treated in Section 3. In Section 4 we look at the relationship between complete atomic regular double Stone algebras and the algebra of all rough subsets of an approximation space.

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2. Perfect extensions

Throughout the paper **A** and **B** will denote regular double Stone algebras. Let $J(\mathbf{A})$ denote the set of join irreducible elements of **A** and let $At(\mathbf{A})$ denote the set of atoms of **A**.

DEFINITION 2.1. For regular double Stone algebras **A** and **B**, **B** is a perfect extension of **A** (also called a canonical extension) if

(i) **A** is a subalgebra of **B** and **B** is complete and atomic;

(ii) (compact) whenever $x_i \in A$ *(all* $i \in I$*)* and $\sum_{i \in I} x_i = 1$, there exist a finite subset J of I such that $\sum_{i \in J} x_j = 1$.

(iii) (separation) whenever $u, v \in J(B)$ with $v \nleq u$, there exist $a \in A$ with $a \geq u$ and $v \nleq a$.

The conditions above reduce to those of Jónsson and Tarski [7] when **A** and **B** are BA's. We begin our development with a few preliminary results about the structure of complete atomic regular double Stone algebras.

LEMMA 2.2. *Suppose* $\mathbf{B} = (\mathbf{C}, F)$ *is a regular double Stone algebra. Then* $x = (a, b) \in B$ *is an atom of* **B** *if, and only if.*

(i) $a \in At(\mathbf{C})$ *and* $b \in \{0, a\}$, *and*

(ii) $b = 0$ *iff* $a' \in F$.

Moreover, **B** *is atomic if and only if* **C** *is atomic.*

Proof. Assume that $x = (a, b) \in B$ is an atom of **B**. If $a \notin At(\mathbf{C})$, there exist $0 < a_1 < a$ and $y = (a_1, a_1 \cdot b) \in B$. Thus, $y < x$ which contradicts $x \in At(\mathbf{B}$; hence $a \in At(\mathbf{C})$. Since $a \geq b$, we have $b \in \{0, a\}$ which shows (i) holds. (ii) holds since $x = (a, 0) \in B$ is equivalent to $a' \in F$. The rest of the proof is straightforward. □

LEMMA 2.3. *Suppose* $\mathbf{B} = (\mathbf{C}, F)$ *is a regular double Stone algebra. Then* **B** *is complete if, and only if,* **C** *is complete and* F *is a principal filter on* **S**. *Moreover, if* **B** *is complete and* $x_i = (a_i, b_i) \in B$ *for all* $i \in I$, *then the following hold*:

(i) $\sum_I x_i = (\sum_I a_i, \sum_I b_i)$,
(ii) $\prod_I x_i = (\prod_I a_i, \prod_I b_i)$,
(iii) $(\sum_I x_i)^* = \prod_I x_i^*$,
(iv) $(\prod_I x_i)^+ = \sum_I x_i^+$,
(v) $(\sum_I x_i)^+ = \prod_I x_i^+$,
(vi) $(\prod_I x_i)^* = \sum_I x_i^*$.

Proof. Suppose **B** is complete. Since the map $x \mapsto x^{**}$ is a closure operator on **B** and $\mathbf{C} \cong C(\mathbf{B})$, **C** is complete (cf., Szasz [13], p. 69). Now, we show that F is principal. Let $(c, d) = \sum \{(a, 0) : a' \in F\}$ whose join exist by completeness. Then $d + c' \in F$. Since $a \le c$ for all $a' \in F$, it follows that c' is a lower bound of F in **C**. Assume to the contrary that F is not a principal filter. Then there exist $e \in F$ with $e < d + c'$. Note that $d \neq 0$ since c' is a lower bound of F. Consider $d \cdot e$. If $d \cdot e = d$ (i.e., $d \le e$), then since $c' \le e$, we obtain $d + c' \le e$ which is a contradiction. Thus, $d \cdot e < d$. It is easy to verify that $(c, d \cdot e) \in B$. Thus,

$$(a, 0) \le (c, d \cdot e) < (c, d)$$

for all $a' \in F$ which contradicts the choice of (c, d). Thus, F is principal; say $F = [f)$. Note that $(f', 0) \ge (a, 0)$ for all $a' \in F$. Thus, $(c, d) \le (f', 0)$ which gives $d = 0$ and $c = f'$. To prove (i) use De Morgan's Law, the infinite distributivity in C, and $a_i' + b_i \in F = [f)$ for all $i \in I$ to obtain

$$\left(\sum_I a_i\right)' + \sum_I b_i = \prod_I a_i' + \sum_I b_i = \prod_I \left(a_i' + \sum_I b_i\right) \ge \prod_I (a_i' + b_i) \in F.$$

Thus, $(\sum a_i, \sum b_i) \in B$ and is clearly the least upper bound of $\{x_i : i \in I\}$. The proof of (ii) is similar. Statements (iii)–(vi) are routine as well as the converse. □

COROLLARY 2.4. *If* $\mathbf{B} = (\mathbf{C}, F)$ *is a complete atomic regular double Stone algebra, then* **B** *is completely distributive.*

Proof. Immediate from 2.3(i), 2.3(ii) and the fact C is completely distributive, cf., Grätzer [5], p. 116. □

Part (i) of the lemma below was inspired by Katriňák's characterization of the subdirectly irreducible double Stone algebras [9]. Parts (ii) and (iii) characterize the join irreducible elements.

LEMMA 2.5. *Suppose* $\mathbf{B} = (\mathbf{C}, F)$ *is an atomic regular double Stone algebra.*
(i) *If* $x \in At(C(\mathbf{B}))$, *then the relativized algebra* $\mathbf{B}_x = (x]$ *is isomorphic to* $\mathbf{S}_2$ *or* $\mathbf{S}_3$;

(ii) *If* $v \in J(\mathbf{B})$, *then either* $v \in At(\mathbf{B})$ *or* $v = b^{**}$ *for some* $b \in At(\mathbf{B})$;
(iii) *If* $v \in At(\mathbf{B})$, *then* $v, v^{**} \in J(\mathbf{B})$.

Proof. (i) follows easily from Lemma 2.2.

(ii) Suppose $v \notin At(\mathbf{B})$. Then there exist $b \in At(\mathbf{B}), b < v$. Then $b^{**} \in At(C(\mathbf{B}))$ and $v = b^{**} \cdot v + b^{*} \cdot v$. Since $v \in J(\mathbf{B})$ and $b^{**} \cdot v \neq 0, v \leq b^{**}$. By (i), $(b^{**}]$ is isomorphic to $\mathbf{S}_3$ and $b < v$ so $v = b^{**}$.

(iii) is obvious from (i). □

LEMMA 2.6. *Suppose* $\mathbf{B} = (\mathbf{C}, F)$ *is a regular double Stone algebra. Then* $\mathbf{B}$ *is complete and atomic if, and only if,* $\mathbf{B} \cong \prod_{x \in X} \mathbf{B}_x$ *where* $X = At(C(\mathbf{B}))$ *and* $\mathbf{B}_x$ *is the algebra obtained by relativizing* $\mathbf{B}$ *to* $x \in X$. *Moreover, for each* x, $\mathbf{B}_x \cong \mathbf{S}_2$ *or* $\mathbf{B}_x \cong \mathbf{S}_3$ *and* $\mathbf{B}_x \cong \mathbf{S}_3$ *iff* $x \in \{y \in X : y' \in F\}$.

Proof. Define the map ϕ on B into the product for each $b \in B$ and $x \in At(C(\mathbf{B}))$ by

$$\phi_b(x) = x \cdot b.$$

To show ϕ is one-one suppose $a \neq b$ in B. By regularity $a^* \neq b^*$ or $a^+ \neq b^+$. Choose an appropriate atom of $C(\mathbf{B})$ to distinguish ϕ_a from ϕ_b. The remaining properties which verify that ϕ is the desired isomorphism are straightforward. The last statement follows from Lemmas 2.2 and 2.5. □

The following result gives information about the structure of perfect extensions.

LEMMA 2.7. *Suppose* $\mathbf{B} = (\mathbf{C}_1, F_1)$ *a perfect extension of* $\mathbf{A} = (\mathbf{C}, F)$. *Then*
(i) *whenever* $u, v \in J(\mathbf{B})$ *and* $u \cdot v = 0$ *(in particular, if* u *and* v *are distinct atoms of* $C(\mathbf{B})$*), there exist* $a \in C(\mathbf{A})$ *with* $a \geq u$ *and* $a \cdot v = 0$;
(ii) $\mathbf{C}_1$ *is a perfect extension of* $\mathbf{C}$ *and* $F_1 = [f)$ *where* $f = \prod F$ *(in* $\mathbf{C}_1$*)*;
(iii) whenever $x_i \in A$ *(for* $i \in I$*),* $x \in A$, *and* $\sum_{i \in I} x_i \geq x$, *there exist a finite subset* J *of* I *such that* $\sum_{i \in J} x_i \geq x$;
(iv) *whenever* $x_i \in A$ *(for* $i \in I$*),* $x \in A$, *and* $\prod_{i \in I} x_i \leq x$, *there exist a finite subset* J *of* I *such that* $\prod_{i \in J} x_i \leq x$;
(v) *if* $\phi : \mathbf{B} \cong \prod_{x \in X} \mathbf{B}_x$ *as in Lemma* 2.6, *then* ϕ *represents* $\mathbf{A}$ *as a subdirect product of* $\langle \mathbf{B}_x : x \in X \rangle$.

Proof. (i) If $u, v \in J(\mathbf{B})$ and $u \cdot v = 0$, then $u^{**} \cdot v^{**} = 0$. By 2.5(ii) and 2.5(iii) we may assume $u, v \in At(C(\mathbf{B}))$. By 2.1(iii) there exist $a \in A$ with $u \leq a$ and $v \nleq a$. Then $a^{++} \in C(\mathbf{A})$ and $u \leq a^{++}$. Also $v \nleq a^{++}$ which implies $v \cdot a^{++} = 0$ since $v \in At(C(\mathbf{B}))$.

(ii) To show $\mathbf{C}_1$ is a perfect extension of $\mathbf{C}$ it suffices to check the Jónsson–Tarski conditions for BA's. These follow using 2.2, 2.3 and (i). By 2.3, $F_1 = [g) \supseteq F$. Then $f = \prod F$ (in $\mathbf{C}_1$) $\geq g$. Suppose $f > g$. Then there exist $a \in At(\mathbf{C})$ such that $a \leq f$ and $a \cdot g = 0$. Set $v = (a, 0)$. Since $a' \geq g \in F_1$, $v \in B$. Note $v^{**} = (a, a)$. Since $v^{**} \nleq v$, separation implies there exist $x = (c, d) \in A$ such that $x \geq v$ and $v^{**} \nleq x$, i.e., $a \leq c$ and $a \nleq d$. If $d = 0$, then $c' \in F$; thus $c \leq f'$ from which we obtain $a \leq c \leq f'$ contradicting the choice of a. Hence, $d \neq 0$. Therefore $a \cdot d = 0$, i.e., $d \leq a'$. This, together with $a' \geq c'$ and $d + c' \geq f$ (since $d + c' \in F$) yield $a' \geq f$ which contradicts $f \nleq a'$. Thus, $f = g$ as desired.

(iii) and (iv) follow from (ii) and Lemma 2.3.

(v) It suffices to assume $\mathbf{B}_x$ is a 3 element chain and $x = b^{**} > b$ for some $b \in At(\mathbf{B})$. By 2.5(ii), $b, b^{**} \in J(\mathbf{B})$ and $b^{**} \nleq b$; so by separation, 2.1(iii), there exist $a \in A, a \geq b$ and $b^{**} \nleq a$. Thus, $b \leq a \cdot b^{**} < b^{**}$ so $a \cdot b^{**} = b$. Thus, the map $y \mapsto y \cdot yb^{**}$ is onto $\mathbf{B}_x$ as desired. □

In preparation for the main result of this section – the existence and uniqueness of perfect extensions – we describe a subdirect decomposition of a regular double Stone algebra that is equivalent to its natural sheaf representation ([2]).

LEMMA 2.8. *Suppose* $\mathbf{B} = (\mathbf{C}, F)$ *is a regular double Stone algebra. Let* X *denote the set of all ultrafilters on* $\mathbf{C}$ *and* $Y = \{y \in X : y \nsupseteq F\}$. *For each* $x \in X$ *define* θ_x *on* A^2 *by*

$$(a, b)\theta_x(c, d) \quad \text{iff } (a, c \in x \text{ or } a, c \in B\backslash x) \quad \text{and} \quad (b, d \in x \text{ or } b, d \in B\backslash x).$$

Then

(i) θ_x *is a congruence relation on* $\mathbf{A}$ *and* $\bigcap \{\theta_x : x \in X\} = Id_A$;

(ii) $\mathbf{A}_x = A/\theta_x \cong \mathbf{S}_2$ *if* $x \in Y$ *and* $\mathbf{A}_x \cong \mathbf{S}_3$ *if* $x \in Y$;

(iii) *the map* $\phi : \mathbf{A} \to \prod_{x \in X} \mathbf{A}_x$ *defined, for all* $a \in A$ *and* $x \in X$, *by*

$$\phi(a)_x = a/\theta_x$$

represents $\mathbf{A}$ *as a subdirect product of* $\langle \mathbf{A}_x : x \in X \rangle$.

The proof is straightforward. For information about congruence relations on double Stone algebras see [10].

The following result shows that perfect extensions exist and are unique up to isomorphism.

THEOREM 2.9. (i) *If* $\mathbf{A}$ *is a regular double Stone algebra, then there exist a perfect extension* $\mathbf{A}^\sigma$ *of* $\mathbf{A}$.

(ii) *If* $\mathbf{B}$ *is a perfect extension of* $\mathbf{A} = (\mathbf{C}, F)$, *then* $\mathbf{B} \cong \prod_{x \in X} \mathbf{A}_x$ *where* $\langle \mathbf{A}_x : x \in X \rangle$ *is the subdirect representation of* $\mathbf{A}$ *given in Lemma* 2.8.

Proof. (i) Set $\mathbf{A}^\sigma = (\mathbf{C}_1, F_1)$ where $\mathbf{C}_1$ is a perfect extension of $\mathbf{C}$ and $F_1 = [f)$ where $f = \prod F$ (in $\mathbf{C}_1$). Then $\mathbf{A}^\sigma$ is complete and atomic (Lemmas 2.2, 2.3) and contains $\mathbf{A}$ as a subalgebra. Because $\mathbf{C}_1$ is a perfect extension of $\mathbf{C}$ it is easy to verify 2.1(ii) that $\mathbf{A}$ is compact in $\mathbf{A}^\sigma$. To prove the separation condition, 2.1(iii), take $\bar{u}, \bar{v} \in J(\mathbf{A}^\sigma)$ with $\bar{v} \not\leq \bar{u}$. By Lemmas 2.2 and 2.5 five cases can occur when u, v are distinct atoms of $\mathbf{C}_1$: (i) $\bar{u} = (u, u)$ and $\bar{v} = (v, v)$, (ii) $\bar{u} = (u, 0)$ and $\bar{v} = (v, 0)$, (iii) $\bar{u} = (u, 0)$ and $\bar{v} = (v, v)$, (iv) $\bar{u} = (u, u)$ and $\bar{v} = (v, 0)$, and (v) $\bar{u} = (u, 0)$ and $\bar{v} = (u, u)$. Since $\mathbf{C}_1$ is a perfect extension of $\mathbf{C}$ there exist $a \in C$ with $u \leq a$ and $v \not\leq a$ in $\mathbf{B}_1$. Setting $\bar{a} = (a, a)$ we have $\bar{u} \leq \bar{a}$ and $\bar{v} \not\leq \bar{a}$ in cases (i)–(iv). In case (v), $u' \geq f$. First suppose $u' = f = \prod F$. Then, since $u \in At(\mathbf{C}_1)$, $u' \in F$ and $\bar{a} = \bar{u}$ has the required property. Now suppose $u' > f = \prod F$. Then there exist $a' \in F$ such that $u' > a' > f$. Thus, $\bar{a} = (a, 0) \in A$, $\bar{u} \leq \bar{a}$, and $\bar{v} \not\leq \bar{a}$ as desired.

(ii) Suppose $\mathbf{B} = (\mathbf{C}_1, F_1)$ is a perfect extension of $\mathbf{A} = (\mathbf{C}, F)$. Then, by Lemma 2.7(ii), $\mathbf{C}_1$ is a perfect extension of $\mathbf{C}$ and $F \subseteq F_1 = [f)$ where $f = \prod F$. Thus, $|At(\mathbf{B})| = |At(\mathbf{C}_1)| = |X|$ where X is the set of all ultrafilters of $\mathbf{C}$ by Lemma 2.2 and the Jónsson–Tarski construction. By Lemma 2.6, $\mathbf{B} \cong \prod_{x \in At(\mathbf{C}_1)} \mathbf{B}_x$ where $\mathbf{B}_x \cong \mathbf{S}_2$ if $x' \not\geq f$ and $\mathbf{B}_x \cong \mathbf{S}_3$ if $x' \geq f$. Since $\mathbf{C}_1$ is a perfect extension of $\mathbf{C}$, for each $y \in At(\mathbf{C}_1)$, $[y) \cap C = \bar{y}$ is an ultrafilter on $\mathbf{C}$. Let $Y = \{y \in X : y \not\supseteq F\}$. Note that $y' \geq f$ iff $\bar{y} \in Y$. The map $y \mapsto \bar{y}$ gives a bijection between $At(\mathbf{C}_1)$ and X that restricts to a bijection between $\{y \in At(\mathbf{C}_1) : y' \geq f\}$ and Y. Bt Lemma 2.8, $\mathbf{B}_x \cong \mathbf{A}_{\bar{x}}$ for all $x \in At(\mathbf{C}_1)$; thus,

$$\mathbf{B} \cong \prod_{y \in X} \mathbf{A}_y,$$

where $\langle \mathbf{A}_y : y \in X \rangle$ is the representation of $\mathbf{A}$ given in Lemma 2.8. □

3. Extending operators

In this section the Extension Theorem of Jónsson and Tarski ([7], Section 2) is extended from Boolean algebras to regular double Stone algebras. The proof is similar to the one for BAs except for a technical complication due to the use of join irreducible elements in $\mathbf{A}^\sigma$ in place of atoms.

Let $\mathbf{A}$ be a regular double Stone algebra and $\mathbf{A}^\sigma$ its perfect extension from Theorem 2.9. Following Jónsson and Tarski [7] we say that $f : \mathbf{A}^n \to \mathbf{A}$ is an *operator* if it is additive in each argument and that f is *normal* if its value is 0 whenever one of its arguments is 0. This notion is called a *join-hemimorphism* in Goldblatt [4]. A *regular double Stone algebra with operators* is an algebra $\langle \mathbf{A}, f_i \rangle_{i \in I}$ where $\mathbf{A}$ is a regular double Stone algebra and all operations f_i are operators on $\mathbf{A}$. If all the f_is are normal, we say $\langle \mathbf{A}, f_i \rangle_{i \in I}$ is *normal.*

An element $x \in \mathbf{A}^\sigma$ is called *closed* if $x = \prod \{y \in A : y \geq x\}$. Similarly, x is *open* if $x = \sum \{y \in A : y \leq x\}$. Let K denote the set of all closed elements of $\mathbf{A}^\sigma$.

Operations f on $\mathbf{A}$ are extended to $\mathbf{A}^\sigma$ in the following well-known manner.

DEFINITION 3.1. For an n-ary operation f on $\mathbf{A}$ the canonical extension f^σ is defined for all $x \in (A^\sigma)^n$ by

$$f^\sigma(x) = \sum_{x \geq a \in K^n} \prod_{a \leq b \in A^n} f(b).$$

The following are straightforward, cf., 2.2 and 2.3 of [7].

LEMMA 3.2. *For an n-ary operation f on* $\mathbf{A}$ *and* $x \in K^n$, $f^\sigma(x) = \prod_{x \leq b \in A^n} f(b)$.

LEMMA 3.3. *If f is a monotone function from A^n to A (in particular, if f is additive), then $f^\sigma(x) = f(x)$ for all $x \in A^n$.*

Below, we establish an alternative description for f^σ.

DEFINITION 3.4. For an n-ary operation f on $\mathbf{A}$ define g for all $x \in (\mathbf{A}^\sigma)^n$ by

$$g(x) = \sum_{x \geq u \in J(\mathbf{A}^\sigma)^n} f^\sigma(u).$$

Observe that, using the separation condition 2.1(iii), a join irreducible is closed so that, in 3.4, $f^\sigma(u)$ is computed as in 3.2. Using Lemma 2.5 we see that if u is join irreducible and $u \leq \sum_I x_i$ in a complete atomic regular double Stone algebra, then $u \leq x_i$ for some $i \in I$. From this observation it is easy to prove the following (cf., [7], 1.7).

LEMMA 3.5. *g is completely additive.*

The following is the main technical result of this section.

LEMMA 3.6. *If f is additive (in each argument) on $\mathbf{A}^n$ to* $\mathbf{A}$, *then $f^\sigma(x) = g(x)$ for all $x \in A$. From* 3.5 *it follows that f^σ is completely additive.*

Proof. Since each join irreducible element is closed, clearly

$$g(x) = \sum_{x \geq u \in J(\mathbf{A}^\sigma)^n} f^\sigma(u) \leq \sum_{x \geq u \in K^n} f^\sigma(u) = f^\sigma(u).$$

It remains to show

$$f^\sigma(y) \leq g(y) \qquad \text{for all } y \in K^n. \tag{1}$$

For, if (1) holds, then, for $x \in (A^\sigma)^n$, by 3.1, 3.2 and (1)

$$f^\sigma(x) = \sum_{x \geq y \in K^n} f^\sigma(y) = \sum_{x \geq y \in K^n} g(y) \leq g(x).$$

Now, consider $y \in K^n$ and suppose $g(y) < f^\sigma(y)$. Then there exist $u \in J(\mathbf{A}^\sigma)$, $u \leq f^\sigma(y)$ and $u \nleq g(y)$. Let $K_1 = \{a : y \geq a \in J(\mathbf{A}^\sigma)^n\}$.

Case 1. $u \in At(\mathbf{A}^\sigma)$.

Since $u \nleq g(y) = \sum_{a \in K_1} f^\sigma(a)$, for all $a \in K_1$, $u \nleq f^\sigma(a) = \prod_{a \leq z \in A^n} f(z)$. Thus

$$\text{for every } a \in K_1, \text{ there is } z_a \text{ with } a \leq z_a \in A^n \text{ and } u \cdot f(z_a) = 0. \tag{2}$$

We want to construct for $a \in K_1$ elements $\phi_{jp}(a) \in A$ for $j < n, p \leq n$ such that

$$y_j \leq \phi_{jp}(a) \quad \text{if } a \in K_1 \quad \text{and} \quad j < p \leq n; \tag{3}$$

$$a_j \leq \phi_{jp}(a) \quad \text{if } a \in K_1 \quad \text{and} \quad p \leq j < n; \tag{4}$$

$$u \cdot f(\phi_{0p}(a), \phi_{1p}(a), \ldots, \phi_{n-1p}(a)) = 0 \quad \text{if } p \leq n \quad \text{and} \quad a \in K_1. \tag{5}$$

The sequences ϕ_{jp} will be constructed by recursion. For $p = 0$, let $\phi_{j0}(a) = (z_a)_j$ for all $j < n$ and $a \in K_1$. Condition (3) is vacuous and (4), (5) follow from (2). Now, assume $p < n$ and $\phi_{jp}(a)$ is defined for all $j < n$, $a \in K_1$, and condition (3), (4), and (5) hold for this p. For $a \in K_1$ let

$$L_1 = \{b \in K_1 : b_j = a_j \text{ for all } j < n \text{ and } j \neq p\}.$$

For every $c \in J(\mathbf{A}^\sigma)$ with $c \leq y_p$ there is $b \in L$ with $b_p = c$. Hence, by (4), applied to p and b, $c = b_p \leq \phi_{pp}(b)$. Since $y_p = \sum \{c \in J(\mathbf{A}^\sigma) : c \leq y_p\}$,

$$y_p \leq \sum_{b \in L_1} \phi_{pp}(b).$$

Since y_p is closed and $\phi_{bb}(b)$ is open, an easy extension of 2.7(iii) implies there exist a finite $M \subseteq L_1$ such that $y_p \leq \sum_{b \in M} \phi_{pp}(b)$. Now define

$$\phi_{pp+1}(a) = \sum_{b \in M} \phi_{pp}(b) \quad \text{and} \tag{6}$$

$$\phi_{jp+1}(a) = \prod_{b \in M} \phi_{jp}(b) \qquad \text{for all } j < n, j \neq p. \tag{7}$$

We must verify (3), (4), and (5) with $p + 1$ in place of p. Condition (3) holds by (6), the induction hypothesis, and (7); (4) follows from (7) and the induction hypothesis. To see (5) use (6), the assumption that f is additive, and (7) to obtain

$$\begin{aligned} &u \cdot f(\phi_{0p+1}(a), \ldots, \sum_{p \in M} \phi_{pp}(b), \ldots) \\ &\quad = \sum_{b \in M} u \cdot f(\phi_{0p+1}(a), \ldots, \phi_{pp}(b), \ldots, \phi_{n-1p+1}(a)) \\ &\quad \leq \sum_{b \in M} u \cdot f(\phi_{0p}(b), \ldots, \phi_{pp}(b), \ldots, \phi_{n-1p}(b)) = 0. \end{aligned}$$

This completes the induction. Now, for $a \in K_1$, define $z \in A^n$ by

$$z_j = \phi_{jn}(a)$$

for all $j < n$. From (3) it follows that $y \leq z$ and from (5), $0 = u \cdot f(z)$. This contradicts the choice of $u : 0 \neq u \leq f^\sigma(y)$ while $u \cdot f^\sigma(y) \leq u \cdot f(z) = 0$.

Case 2. $u \in J(\mathbf{A}^\sigma) \backslash At(\mathbf{A}^\sigma)$.

From Lemma 2.5 it follows that $u \in At(C(\mathbf{A}^\sigma))$. As in Case 1, for every $a \in K_1$, $u \nleq f^\sigma(a) = \prod_{a \leq z \in A^n} f(z)$. Thus,

$$u \nleq f^\sigma(a)^{++} = \prod_{a \leq z \in A^n} f(z)^{++}.$$

Since $u \in At(C(\mathrm{A}^\sigma))$ there exist u_a with

$$a \leq u_a \in A^n \quad \text{and} \quad u \cdot f(u_a)^{++} = 0. \tag{8}$$

We proceed as in Case 1 except that we modify condition (5) slightly. We construct for $a \in K_1$ elements $\phi_{jp}(a) \in A$ for all $j < n, p \leq n$ such that (3), (4), and

$$u \cdot f(\phi_{0p}(a), \ldots, \phi_{n-1p}(a))^{++} = 0 \qquad \text{if } p \leq n \text{ and } a \in K_1. \tag{9}$$

The sequence of ϕ_{jp}s is constructed as in Case 1 using the u_as in place of the z_as to start the induction. In the induction step $\phi_{jp+1}(a)$ is defined as in (6) and (7). The elements will clearly satisfy conditions (3) and (4) as before. The proof of (9) is similar to that in Case 1 again using the assumption that f is additive and the dual Stone property $(u+v)^{++} = u^{++} + v^{++}$. Namely,

$$\begin{aligned} &u \cdot f(\phi_{0p+1}(a), \ldots, \sum_{p \in M} \phi_{pp}(b), \ldots)^{++} \\ &\quad = \sum_{b \in M} u \cdot f(\phi_{0p+1}(a), \ldots, \phi_{pp}(b), \ldots, \phi_{n-1p+1}(a))^{++} \\ &\quad \leq \sum_{b \in M} u \cdot f(\phi_{0p}(b), \ldots, \phi_{pp}(b), \ldots, \phi_{n-1p}(b))^{++} = 0. \end{aligned}$$

This completes the induction. Now, for $a \in K_1$ define $z \in A^n$ by $z_j = \phi_{jn}(a)$ for all $j < n$. From (3) it follows that $y \leq z$ and from (9), $0 = u \cdot f(z)^{++}$.

These properties produce the desired contradiction since $u \in At(C(\mathbf{A}^\sigma))$, $u \leq f^\sigma(x)$, and $y \leq z$ gives $u \leq f(y)^{++} \leq f(z)^{++}$ contrary to $u \cdot f(z)^{++} = 0$. Since both Case 1 and Case 2 produce contradictions, $f^\sigma(y) \leq g(y)$; so (1) holds as desired. □

From Lemmas 3.3 and 3.6,

THEOREM 3.7. *Every n-ary operator f on a regular double Stone algebra $\mathbf{A}$ has a completely additive extension f^σ to $\mathbf{A}^\sigma$. f^σ is normal if f is.*

As in the case of Boolean algebras (cf., 2.5 of [7]) it is easy to show that f^σ is the largest possible extension of f to a completely additive function on $\mathbf{A}^\sigma$.

LEMMA 3.8. *If f is an operator on A^n and g is a completely additive function from $(A^\sigma)^n$ to A^σ such that $g(x) \leq f(x)$ for all $x \in A^n$, then $g \leq f^\sigma$.*

The following result combines 2.9 and 3.7.

THEOREM 3.9 (The Extension Theorem). *For every regular double Stone algebra with operators $\mathbf{A}$ there exist a complete, completely distributive atomic perfect extension $\mathbf{A}^\sigma$ of $\mathbf{A}$ with completely additive operators. Moreover, if $\mathbf{A}$ is normal, then so is $\mathbf{A}^\sigma$.*

4. Connections with rough sets

An aspect of the theory of perfect extensions [7] that lends itself to the construction of useful examples is the connection between perfect extensions of Boolean algebras with operators and complex algebras of relational systems. In this section we establish a connection between the perfect extension in Section 2 and "complex" algebras of rough sets of an approximation space.

Rough sets were introduced by Pawlak [11]. We follow the lattice theoretic approach due to Iwinski [6]. A pair $\mathbf{U} = \langle U, \theta \rangle$ that consists of an equivalence relation θ on a nonempty set U is called an *approximation space*. Every $X \subseteq U$ has a lower approximation $\underline{X}$ and an upper approximation $\bar{X}$ defined by

$$\underline{X} = \bigcup \{\theta x : \theta x \subseteq X\} \quad \text{and}$$

$$\bar{X} = \bigcup \{\theta x : x \in X\}.$$

A *rough subset* of $\mathbf{U}$ is a pair $(\underline{X}, \bar{X})$ where $X \subseteq U$. The collection of all rough subsets of U is denoted by $Sb_R(\mathbf{U})$ and the algebra of all rough subset of $\mathbf{U}$ is $\mathbf{P}_R(\mathbf{U}) = \langle Sb_R(U), \vee, \wedge, {}^*, {}^+, 0, 1\rangle$ where $0 = (\emptyset, \emptyset)$, $1 = (U, U)$, $\vee$ and $\wedge$ are defined coordinatewise, $(\underline{X}, \bar{X})^* = (U \backslash \bar{X}, U \backslash \bar{X})$, and $(\underline{X}, \bar{X})^+ = (U \backslash \underline{X}, U \backslash \underline{X})$. It was shown in [12] that $\mathbf{P}_R(\mathbf{U})$ is a Stone algebra.

The characterization of complete atomic regular double Stone algebras in Theorem 4.1 was given in [3], but the proof here is more intrinsic.

THEOREM 4.1. *If* $\mathbf{B}$ *is a complete atomic regular double Stone algebra, then* $\mathbf{B} \cong \mathbf{P}_R(U)$ *for some approximation space* $\mathbf{U}$.

Proof. Let $U = J(\mathbf{B})$ and define θ, for $p, q \in U$ by

$$p\theta q \qquad \text{iff } p^{**} = q^{**}.$$

Clearly $\mathbf{U} = \langle U, \theta \rangle$ is an approximation space. Define $j : B \to Sb(U)$ by $j(b) = \{p \in J(\mathbf{B}) : p \leq b\}$ for all $b \in B$. Observe that

$$\text{whenever } b \in C(B),\ q\theta p \in j(b) \quad \Rightarrow \quad q \in j(b).$$

For $b \in B$ define $\phi(b) = (j(b^{++}), j(b^{**}))$. We want to show that $\phi(b)$ gives the rough set determined by $j(b)$. This is done in (10) and (11) below.

$$j(b^{**}) = \overline{j(b)} \qquad \text{for } b \in B. \tag{10}$$

Using 2.3, $b^{**} = \sum \{p^{**} : p \in j(b)\}$ so $j(b^{**}) = \bigcup \{\theta p : p \in j(b)\} = \overline{j(b)}$ which establishes (10).

$$j(b^{++}) = \underline{j(b)} \qquad \text{for } b \in B. \tag{11}$$

For $b \in B$, we split $j(b)$ into two parts J and K so that $b = \sum J + \sum K$ where $p \in J \Leftrightarrow p^{**} \in j(b)$ and $K = j(b) \backslash J$. Note that $p \in K \Leftrightarrow p \in j(b)$ and $p^{**} \notin j(b)$. Also,

$$\textstyle\sum J = \sum \{p^{**} : p \in K\} \in C(B) \quad \text{and} \tag{12}$$

$$b^{**} = \textstyle\sum J + (\sum K)^{**} \quad \text{and} \quad (\sum J) \cdot (\sum K)^{**} = 0. \tag{13}$$

Now,

$$\underline{j(b)} = J. \tag{14}$$

To see this first observe $\underline{j(b)} = \bigcup \{\theta p : \theta p \subseteq j(b)\} \supseteq J$ since $\theta p \subseteq J \subseteq j(b)$ whenever $p \in J$. On the other hand, if $p \in K$, $\theta p \not\subseteq j(b)$ so $p \notin j(b)$.

$$j(b^{++}) = J. \tag{15}$$

Suppose $(\sum K)^{++} \neq 0$. Then there is $x \in At(C(\mathbf{B}))$ such that $x \leq (\sum K)^{++}$. But $x = q^{**}$ for some $q \in J(\mathbf{B})$ and $q \leq q^{**} \leq (\sum K)^{++} \leq \sum K \leq b$. Hence $q^{**} \in J$ which contradicts (13). Thus, $(\sum K)^{++} = 0$ and, using (12), $b^{++} = (\sum J)^{++} + (\sum K)^{++} = \sum J$. Hence (15) follows. Condition (11) follows from (14) and (15).

From (10) and (11) above we see that $\phi(b) = (j(b^{++}), j(b^{**})) \in Sb_R(\mathbf{U})$. The regularity axiom for $\mathbf{B}$ implies that ϕ is one-one. To see that ϕ is onto $Sb_R(\mathbf{U})$ suppose $S \subseteq T \subseteq J(\mathbf{B})$ where S, T are θ-invariant. Define

$$b = \textstyle\sum \{p^{**} : p \in S\} + \sum \{p : p < p^{**} \in T \backslash S\}.$$

Then $b^{**} = \sum T$ and $b^{++} = \sum S$ so $\phi(b) = (S, T)$ as desired.

It is routine to show that ϕ preserves $+$ and $\cdot$. Consider $*$:

$$\phi(b^{*}) = (j(b^{*++}), j(b^{***})) = (j(b^{*}), j(b^{*})) = \phi(b)^{*}$$

since b^{*} is the complement of b^{**} in $C(\mathbf{B})$. Similarly, $\phi(b^{+}) = \phi(b)^{+}$. This completes the proof that ϕ is the desired isomorphism. $\square$

Theorem 2.9(i) and 4.1 yield the representation result mentioned.

COROLLARY 4.2. *Every regular double Stone algebra is isomorphic to an algebra of rough subsets of an approximation space.*

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