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HYPERSTRUCTURES ASSOCIATED WITH CHARACTER ALGEBRAS AND COLOR SCHEMES[1]

Stephen D. Comer
The Citadel
Charleston, SC 29409, USA
Email address: comers@citadel.edu

Abstract: It is the purpose of this paper to provide examples of hyperstructures associated with the study of symmetry in physics and chemistry. In applications the symmetries of an object are typically considered as a group; so most of the examples presented are connected to constructions from group theory. In particular, we show that a natural hypergroup is associated with every character algebra. This unifies several classical hypergroup constructions. We also show how certain edge colorings of graphs give raise to hypergroups with special properties.

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1. Introduction.

The study of symmetry is closely related to the theory of groups and its extensions. Symmetry groups have been widely applied in chemistry [12], crystallography [17], and solid-state physics [1]. In these areas distinctions are usually made between *symmetric objects* (planes, axes, etc) and *symmetric operations* (reflections, rotations, etc) which leave an object invariant. In this paper *symmetry* will refer to symmetric operations. Many applications of groups involving symmetries have involved coset decompositions, double coset decompositions, decompositions into conjugacy classes, and group characters. Below we introduce a hypergroup structure on each of these "spaces". This paper is intended to be a source of ideas about constructions to show a non-specialist that hypergroups are fairly naturally associated with objects with which they have familiarity.

Section 3 contains a specific result. Namely, we show that a C-algebra in the sense of Y. Kawada [18] gives raise to a special hypergroup called a quasi-cannonical hypergroup by Bonansinga and Corsini [4] and a polygroup in [6].

A *polygroup* is a system $\langle M, \bullet, e\rangle$ where $e \in M$, $\bullet$ assigns a nonempty subset of M to each pair of elements in M, and the following axioms hold for all $x, y, z \in M$:

(1) for each x there exists a unique $x^{-1} \in M$ such that $e \in x \bullet x^{-1}$ and $e \in x^{-1} \bullet x$,

(2) $e \bullet x = \{x\} = x \bullet e$,

(3) $x \in y \bullet z$ implies $y \in x \bullet z^{-1}$ and $z \in y^{-1} \bullet x$,

(4) $(x \bullet y) \bullet z = x \bullet (y \bullet z)$.

Sections 4 and 5 deal with connections between hypergroups and colorings of designs or graphs. In Section 4 we describe the principal coloring associated with a subgroup of a group and in Section 5 we look at polygroups derived from an edge coloring of a complete graph that satisfies a regularity condition.

Throughout, we illustrate constructions using the dihedral group D_4. This group is generated by a counter-clockwise rotation r of 90° and a horizontal reflection h. The group consists of the following 8 symmetries:

$$\{1 = r^0, r, r^2 = s, r^3 = t, h, hr = d, hr^2 = v, hr^3 = f\}.$$

Figure 1 below shows the actions of r, h, v, d, and f.

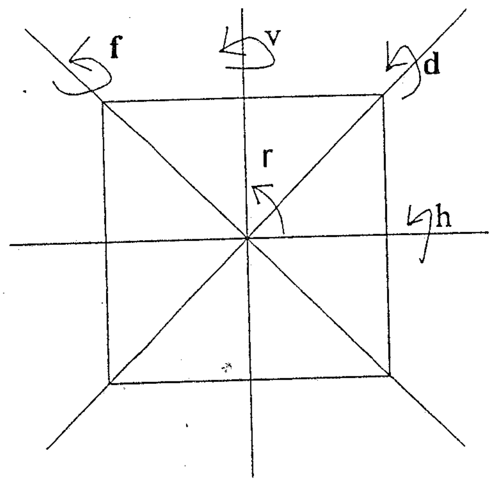


Figure 1 : D_4

2. Classical hypergroup constructions.

Coset Hypergroups. For a group G and a subgroup H let G/H denote the collection of all right cosets $Hg = \{hg: h \in H\}$ for g in G. The system $\langle G/H, \bullet, H\rangle$ is a hypergroup where

$$(Hg_1) \bullet (Hg_2) = \{Hg : g \in g_1 H g_2\}.$$

The system G/H is called a *D-hypergroup* and is the primary example of a cogroup. These systems have been studied by Eaton [15], Krasner [19], and Utumi [25]. Recently Y. Sureau [24] has characterized all cogroups by nested triples of permutation groups.

As an example of the construction consider the subgroup $H = \{1, h\}$ of the group D_4. The right coset decomposition of D_4 with respect to H is

$$\{1, h\} \cup \{r, d\} \cup \{s, v\} \cup \{t, f\} .$$

If two cosets are multiplied elementwise the result may not be a single coset. For example, consider the product $(Ht)(Hr)$ in D_4. This part of the group table for D_4 is displayed below.

D_4	·	r	d	·
·	·	·	·	·
t	·	1	v	·
f	·	h	s	·

We see that $(Ht)(Hr) = H \cup Hs$. Thus, in the operation table for D_4/H the product $Ht \bullet Hr$ is recorded as the two elements H, Hs. Using 1, r, s, and t as coset representatives, the complete operation table for the cogroup D_4/H becomes

	H	Hr	Hs	Ht
H	H	Hr	Hs	Ht
Hr	Hr, Ht	Hs, H	Ht, Hr	H, Hs
Hs	Hs	Ht	H	Hr
Ht	Ht, Hr	H, Hs	Hr, Ht	Ht, H

In Section 4 we show how cogroups such as the one above derived from coset decompositions represent an algebra of colors associated with the subgroup.

Double coset algebras. The unit element of a cogroup is normally a one-sided scalar, not a 2-sided scalar. This is overcome by looking at double cosets. A double coset HgH of a subgroup H of a group G is the collection of all distinct elements h_1gh_2 where h_1 and h_2 range over H. The double cosets of H give a decomposition of G. Let $G//H$ denote the collection of all double cosets of G with respect to H. A *double coset algebra* is a hypergroup $\langle G//H, \bullet, H\rangle$ where

$$(Hg_1H)\bullet(Hg_2H) = \{Hg_1hg_2H : h \in H\}.$$

The system $G//H$ is a example of a polygroup or quasi-cannonical hypergroup.

If $H = \{1, h\}$ is the subgroup of D_4 considered earlier, the double coset decomposition is

$$\{1, h\} \cup \{r, d, t, f\} \cup \{s, v\}\ .$$

As with the elementwise product of two cosets the elementwise product of double cosets can also produce more than one double coset. Consider the product $(HrH)(HrH)$. The relevant part of the group table of D_4 is displayed below.

D_4	·	r	d	t	f	·
·	·	·	·	·	·	·
r	·	s	h	1	v	·
d	·	v	1	h	s	·
t	·	1	v	s	h	·
f	·	h	s	v	1	·
·	·	·	·	·	·	·

We see that $(HrH)(HrH) = H \cup HsH$; thus the polygroup product $HrH \bullet HrH$ is the two elements H, HsH. The full table for the polygroup $D_4//H$ is given below. Notice that the unit element H is a scalar element.

	H	HrH	HsH
H	H	HrH	HsH
HrH	HrH	H, HsH	HrH
HsH	HsH	HrH	H

Conjugacy class polygroups. In dealing with a symmetry group two symmetric operations belong to the same class if they represent the same map with respect to (possibly) different coordinate systems where one coordinate system is converted into the other by a member of the group (cf., [12]). In the language of group theory this means elements a, b in a symmetry group G belong to the same class if there exist a $g \in G$ such that $a = gbg^{-1}$, i.e., a and b are conjugate. The collection of all conjugacy classes of a group G is denoted by $\bar{G}$ and the system $\langle \bar{G} \bullet, \{e\} \rangle$ is a polygroup where e is the identity of G and the product $A \bullet B$ of conjugacy classes A and B consists of all conjugacy classes contained in the elementwise product AB. This hypergroup was recognized by Campaigne ([5]) and by Dietzman ([14]).

In the case of D_4 there are 5 conjugacy classes: $\{1\}$, $\{s\}$, $\{r,t\}$, $\{d,f\}$, and $\{h,v\}$. Let us denote these classes as $C_1,\ldots, C_5$ respectively. Then the polygroup $\bar{D}_4$ is

	C_1	C_2	C_3	C_4	C_5
C_1	C_1	C_2	C_3	C_4	C_5
C_2	C_2	C_1	C_3	C_4	C_5
C_3	C_3	C_3	C_1,C_2	C_5	C_4
C_4	C_4	C_4	C_5	C_1,C_2	C_3
C_5	C_5	C_5	C_4	C_3	C_1,C_2

As a sample of how to calculate the table entries consider $C_3{\cdot}C_3$. To determine this product compute the elementwise product of the conjugacy classes $\{r,t\}\{r,t\} = \{s,1\} = C_1 \cup C_2$. Thus, $C_3{\cdot}C_3$ consists of the two conjugacy classes C_1, C_2.

Character polygroups. Closely related to the conjugacy classes of a finite group are its characters. Let $\hat{G} = \{\chi_1, \chi_2, \cdots, \chi_k\}$ be the collection of irreducible characters of a finite group G where χ_1 is the trivial character. The character polygroup $\hat{G}$ of G is the system $\langle \hat{G}, \bullet, \chi_1 \rangle$ where the product $\chi_i \bullet \chi_j$ is the set of irreducible components in the elementwise product $\chi_i\chi_j$. The system $\hat{G}$ was investigated by R. Roth [21] who considered a duality between $\hat{G}$ and $\overline{G}$.

Before calculating $\hat{D}_4$ we need to know the 5 irreducible characters of the dihedral group D_4. These are given by the following character table.. (Since characters are constant on conjugacy classes it is usual to list only the conjugacy classes across the top of the table.)

	C_1	C_2	C_3	C_4	C_5
χ_1 :	1	1	1	1	1
χ_2 :	1	1	-1	1	-1
χ_3 :	1	1	-1	-1	1
χ_4 :	1	1	1	-1	-1
χ_5 :	2	-2	0	0	0

We illustrate the calculation of the polygroup product of two characters by considering $\chi_5 \bullet \chi_5$. The pointwise product of χ_5 with itself yields the following (non-irreducible) character:

	C_1	C_2	C_3	C_4	C_5
$\chi_5\chi_5$:	4	4	0	0	0

This character can be written as a sum of irreducible characters in exactly one way: $\chi_5\chi_5 = \chi_1 + \chi_2 + \chi_3 + \chi_4$. This is indicated by the entry in the lower right hand corner of the polygroup table for $\hat{D}_4$. In general the polygroup product of two characters $\chi_i \bullet \chi_j$ tells which irreducible characters are in the product $\chi_i \chi_j$ but not the multiplicity. Using i in place of the character χ_i the polygroup $\hat{D}_4$ is

	1	2	3	4	5
1	1	2	3	4	5
2	2	1	4	3	5
3	3	4	1	2	5
4	4	3	2	1	5
5	5	5	5	5	1,2,3,4

3. A construction from C-algebras.

The above constructions of hypergroups based on double cosets, conjugacy classes, and characters all make use of the fact that when two objects of a type under consideration (i.e., double cosets, characters, etc) are multiplied in the natural way, the product is uniquely composed of objects in the set (perhaps with repetition). The polygroup binary operation is a type of convolution which tells which objects are in the product but not the multiplicity. We formalize this idea by showing that every C-algebra gives raise to a polygroup. The notion of C-algebra (or character algebra) presented here is due to Y. Kawada [18] (see also [2]) except that the commutativity requirement has been weakened.

A *C-algebra* is a classical algebra A over the complex numbers together with a basis $X = \{x_0, \cdots, x_d\}$ for A (as a complex linear space) such that

(C_1) A is an algebra and $x_i \bullet x_j = \sum_k p_{ij}^k x_k$ for all i, j,

(C_2) A has an identity element $e = x_0$, i.e., $p_{0j}^k = \delta_{jk} = p_{j0}^k$,

(C_3) every p_{ij}^k is a real number,

(C_4) there exist a permutation $i \to i'$ (for $i = 0, \ldots, d$) such that $(i')' = i$ and

$p_{ij}^{k} = p_{j'i'}^{k'}$,

(C$_5$) $p_{ji}^{0} = p_{ij}^{0} = \kappa_i \delta_{ij'}$ with $\kappa_i > 0$ for all i, j, and

(C$_6$) the map $x_i \mapsto \kappa_i$ induces a linear representation of A.

The condition (C$_4$) implies that the map $x_i \mapsto x_{i'}$ extends to an anti-automorphism of A. A C-algebra is *commutative* if $p_{ij}^{k} = p_{ji}^{k}$ for all i, j, k. The lemma below summarizes a few elementary facts.

Lemma. (1) $0' = 0$,

(2) $\kappa_0 = 1$,

(3) $\kappa_i = \kappa_{i'}$,

(4) $\kappa_k p_{ij}^{k} = \kappa_i p_{kj'}^{i} = \kappa_j p_{i'k}^{j}$.

Proposition. *Every C-algebra A with basis X such that the parameters p_{ij}^{k} are all non-negative (the Kreim condition) determines a polygroup $Pg(A)$ = $\langle X, \bullet, e\rangle$ where $x_i \bullet x_j = \{x_k : p_{ij}^{k} \neq 0\}$ and $x_i^{-1} = x_{i'}$ for all i, j.*

Proof. Since $x_i \bullet x_{i'} = \sum_k p_{ii'}^{k} x_k = \kappa_i x_0 + \cdots$ and $\kappa_i > 0$, it is clear that $x_0 \in x_i \bullet x_{i'}$. If $x_0 \in x_i \bullet x_{j'}$, $p_{ij}^{0} \neq 0$ which implies $j = i'$ by (C$_5$). Similarly, $x_{i'}$ is the only y such that $x_0 \in y \bullet x_i$, so axiom (1) holds. Axiom (2) follows from the C-algebra property (C$_2$) and axiom (3) from Lemma (4). For (4) notice that $x_u \in (x_i \bullet x_j) \bullet x_k \Leftrightarrow p_{ij}^{v} p_{vk}^{u} \neq 0$ if and only if $p_{ij}^{v} p_{vk}^{u} \neq 0$ for some v and similarly, $x_u \in x_i \bullet (x_j \bullet x_k)$ if and only if $p_{iv}^{u} p_{jk}^{v} \neq 0$ for some v. The associative law for $Pg(A)$ follows from the equality $\sum_v p_{ij}^{v} p_{vk}^{u} = \sum_v p_{iv}^{u} p_{jk}^{uv}$ (a consequence of (C$_1$)) and the Krein condition. □

Examples of C-algebras not only include the situations mentioned earlier, but also the adjacency algebras of association schemes ([2]), S-algebras over finite groups ([3]), and centralizer algebras of homogeneous coherent configurations ([16]). The centralizer algebra of a coherent configuration is called a cellular algebra in the work of Weisfeiler ([27]) on the graph isomorphism problem.

4. Color symmetries

In Section 2 we showed how a coset decomposition gave raise to a cogroup. Coset decompositions are related to the study of color symmetries of designs and crystals. The original influence for this was the work of Shubnikov [23] on crystallography. The ideas illustrated here follow the treatment given by Roth [22]. Below we illustrate how a transitive symmetric coloring is associated with a coset decomposition. Since every transitive symmetric coloring is equivalent to one constructed this way, by the construction in Section 2, every transitive symmetric coloring has an associated cogroup. We first define color symmetries and symmetric colorings, then give their construction.

Suppose G is the group of symmetries of a design that has its regions colored (with finitely many colors). An element $g \in G$ is a *color symmetry* if it induces a permutation of the set of colors, i.e., whenever g maps a region colored i onto one colored j, then g must map all regions colored i onto regions colored j. The color symmetries form a subgroup of the group G of all symmetries of a design. If every element of G is a color symmetry, the assignment of colors to the regions of the design is called a *symmetric coloring*. We are interested in *transitive* symmetric colorings, i.e., symmetric colorings such that any color can be mapped into any other by a suitable element of G.

In order to set up a correspondence between the regions of a design and the elements of the symmetric group G of the design a *sequence of fundamental regions* of the design is selected. This is a collection of disjoint regions such that for any two regions in the collection there is a unique symmetry in G that maps one onto the other. Fixing such a sequence of regions allows a unique element of the symmetry group G to be assigned to each region. To start the assignment select an arbitrary region Ω to correspond to the identity element of G. Then, for each fundamental region R, there is a unique $g \in G$ which maps Ω onto R. Assign the group element label "g" to the region R.

The "cross" in Figure 2 illustrates the labeling. This design has 8 fundamental regions and the dihedral group D_4 as its symmetry group. Selecting the region labeled Ω as the initial region, the other regions correspond to the group elements indicated.

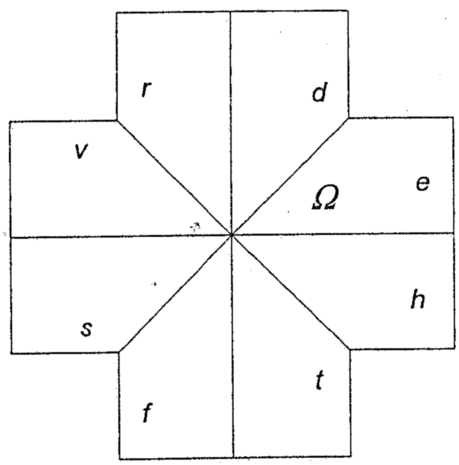


Figure 2: Region labels

If a design has a transitive symmetric coloring and H is the stablizer subgroup of the color of the Ω region, there is a one-one correspondence between the right cosets of H and the set of colors. This leads us to the *principal coloring* of the design associated with a subgroup H of finite index in G. Choose a set of colors in one-one correspondence with the elements of the cogroup G/H; say, use the colors 1,...,n where $G = H \cup Hx_2 \cup \ldots \cup Hx_n$ (with $x_1 = e$). Assign the color i to all regions labeled with elements in the coset Hx_i. This gives a transitive symmetric coloring of the design.

As an example, consider the subgroup $H = \{1,h\}$ of the dihedral group D_4. Assign $\{1,h\}$ ⇒ white, $\{r,d\}$ ⇒ red, $\{s,v\}$ ⇒ blue, and $\{t,f\}$ ⇒ green. The symmetric coloring of the design from H is given in Figure 3 below.

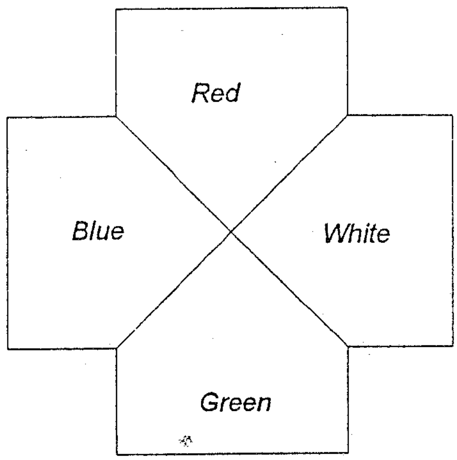


Figure 3: Coloring from H

Roth ([22]) shows that every transitive symmetric coloring (with the same sequence of fundamental regions) is equivalent to one obtained by the process above. This and much more is treated in [22]. The discussion above shows that a cogroup can be assigned to every transitive symmetric coloring of a design, namely, G/H if the coloring corresponds to the subgroup H of G.

	W	R	B	G
W	W	R	B	G
R	R, G	B, W	G, R	W, B
B	B	G	W	R
G	G, R	W, B	R, G	G, W

Using the obvious abbreviations W, R, B, and G for the colors white, red, blue, and green respectively, the table above is the cogroup G/H assigned to the coloring in Figure 3. It is the same cogroup given in Section 2:

5. Color algebras

An *association scheme* [2] is a partition (or coloring) of the edges of a complete graph V in a very regular way. As mentioned in Section 3 the adjacency algebra of the collection of relations of an association scheme is a C-algebra. The notion of a color scheme ([6], [7], [9]) given below is a generalization of the idea of an association scheme and will have a natural polygroup, called a color algebra, associated with it. In the definition below "|" denotes relation composition.

Suppose C is a set (of colors) with a distinguished element $0 \in C$. (Think of 0 as a 'neutral' color.) Suppose $i : C \to C$ with i^2 equal the identity on C and $i(0) = 0$. (Think of $i(c)$ as the color 'opposite' to color c.) A *C-color scheme* is a system $<V, \{C_a : a \in C\}>$ such that $C_a \subseteq V \times V$ for all $a \in C$ and

(1) $\{C_a : a \in C\}$ is a partition of $V \times V$ and C_0 is the identity on V,

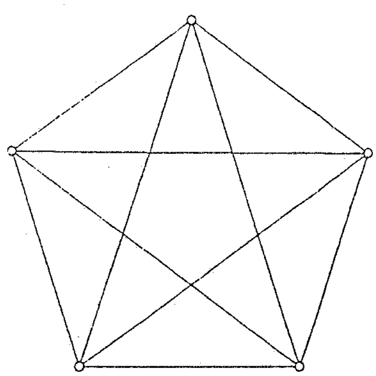
(2) $C_{i(a)} = C_a^{\smile}$ (the converse of C_a) for each $a \in C$,

(3) for every a, b, c $\in C$, $C_c \cap (C_a | C_b) \neq \varnothing$ implies $C_c \subseteq C_a | C_b$.

Condition (1) means that colors are assigned to all edges of a complete directed graph and Condition (2) means that for every edge the color assigned to the edge in the reverse direction is uniquely determined by the color on the original edge. Condition (3) means that whenever some c-colored edge is part of a triangle whose other sides are colored a and b, then every c-colored edge is part of such a triangle. Association schemes and homogeneous coherent configurations are examples of color schemes that satisfy the stronger property (3'), given below, which says that if one c-colored edge is part of k triangles whose other edges are colored a and b, then every c-colored edge is a part of k such triangles.

(3') for every a, b, c $\in C$ and x, y $\in V$, the number of elements in the set $\{z\in V:(x,z)\in C_a \text{ and } (z,y)\in C_b\}$ is independent of the edge (x,y) $\in C_c$.

The following 5 point example has two colors (solid and dash) in addition to the neutral color.



Color 0 = Neutral (points)

Color 1 = Solid line

Color 2 = Dashed line

Figure 4

The *color algebra* of a C-color scheme $<V, \{C_a : a\in C\}>$ is the system $< C, *, 0 >$ based on the set of colors C with * defined for a, b $\in C$ by

$$a*b = \{c\in C : C_c \subseteq C_a|C_b\}.$$

It is easy to see that a color algebra is a polygroup and that the inverse of a color c is the color i(c). As an example of how the product operation works we form the color algebra of the undirected graph in Figure 4 above.

	0	1	2
0	0	1	2
1	1	0,2	1,2
2	2	1,2	0,1

Let 0 denote the neutral color, 1 the solid line color, and 2 the dotted line color. To compute 1*1 look at the composition $C_1|C_1$. It is $C_0 \cup C_2$. Likewise, we see that $C_1|C_2 = C_1 \cup C_2$ and $C_2|C_2 = C_0 \cup C_1$. With these calculations we see that the color algebra of the graph in Figure 4 has the operation table above:

For the record there are exactly 10 three element polygroups ([20], [9]).

A polygroup is called *chromatic* if it is isomorphic to the color algebra of some color scheme. Chromatic polygroups have the nice property that they are exactly the polygroups which have a faithful representation as a regular polygroup of generalized permutations. In other words, a strong form of the Cayley representation holds for these hypergroups. (See, Comer [9].) For another representation of hypergroups by generalized permutations, see [26].

We conclude with a few remarks about other applications of polygroups. Polygroups are closely related to the study of the theory of relations. In [7] an alternate to the relational calculus is proposed which is based on an extension of the notion of polygroup. The polygroupoid notion introduced there is a partial polygroup-like system with a set of identity elements instead of a single identity. To define these systems it is convenient to make the inverse operation explicit. In more detail, a *polygroupoid* is a partial hyperalgebraic structure $<A, \circ, I, ^{-1}>$ where $\circ$ is a partial binary hyperoperation on A, $I \subseteq A$, and $^{-1}$ is a unary operation on A such that the following axioms hold for all $x, y, z \in A$:

(i) $(x \circ y) \circ z = x \circ (y \circ z)$

(ii) $x \circ I = x = I \circ x$

(iii) the formulas $x \in y \circ z$, $y \in x \circ z^{-1}$, and $z \in y^{-1} \circ x$ are equivalent.

Condition (i) should be interpreted as saying that if either side is non-empty, then both sides are non-empty and the sets are equal.

Polygroupoids correspond to the atom structures of systems of relations. In a sense polygroupoids (and the special case, polygroups) are a way to combinatorially classify types of relations. One effort in this direction is presented in [8]. Another example deals with the association schemes associated with regular trees by Delsarte ([13]). In [10] the color algebras associated with regular trees are completely described. In [11] polygroupoids are generalized further to partial multi-valued loops. These systems arise in the representation of atom structures of 3-dimensional cylindric algebras.

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